MATH 571 Comprehensive Notes

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1 Introduction to Systems of Equations (Lecture 1)

In linear algebra, a *system of equations* implies a system of linear equations.

1.1 Example (3 variable, 2 equations)

How does one solve this?

$$\begin{cases} x_1 & -2x_2 & +3x_3 & = 1 \\ x_1 & & -x_3 & = 2 \end{cases}$$

We want all tuples (x_1, x_2, x_3) such that all equations are satisfied. So, take the first equation and subtract the second equation.

$$\begin{cases} x_1 & -2x_2 & +3x_3 & = 1 \\ & 2x_2 & -4x_3 & = 1 \end{cases}$$

This is the *row echelon form* of the system. Now rewrite again.

$$\begin{cases} x_1 & -2x_2 & = 1 - 3x_3 \\ & 2x_2 & = 1 + 4x_3 \end{cases}$$

Finally, back-substitute.

$$x_2 = \frac{1 + 4x_3}{2} = 2x_3 + \frac{1}{2}$$

That's x_2. Now x_1.

$$x_1 = 2x_2 + 1 - 3x_3 = (1 + 4x_3) + 1 - 3x_3 = 2 + x_3$$

The solution of the system is

$$\left\{ \begin{pmatrix} 2 + x_3 \\ \frac{1}{2} + 2x_3 \\ x_3 \end{pmatrix} |x_3 \in \Re \right\}$$

Or also written as

$$\left\{ \begin{pmatrix} 2 \\ \frac{1}{2} \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} |t \in \Re \right\}$$

Definition (Equivalence of Systems) Two systems are *equivalent* if they have the same set of solutions.

Theorem (Solution sets for two variable systems) A two-variable system of equations might have no solutions, a unique solution, or infinitely many solutions. (Did a pretty self-evident "picture proof" in class.)

1.2 Justification of Notation (Lecture 4)

Why is a system described by $A\vec{x} = \vec{b}$? Consider the following system.

$$\begin{cases} x_1 & -x_2 & +x_3 & = 1 \\ x_1 & & +x_3 & = 2 \end{cases}$$

Let this matrix be represented by the following.

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Now checking the equivalence.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A \cdot \vec{x} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \cdot x_1 + (-1) \cdot x_2 + 1 \cdot x_3 \\ 1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 + x_3 \\ x_1 + x_3 \end{pmatrix}$$

Clearly, this is the same system. We can also write further.

$$A \cdot \vec{x} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus, the full system can be written as the following.

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

2 Matricies (Lecture 2–3)

Every system of equations has a corresponding matrix representation. For example, consider this system.

$$\begin{cases} x_1 & & +x_3 & & +x_5 & & = 7 \\ & x_2 & & & & +x_6 & = 1 \\ x_1 & & & & +x_5 & & = 3 \end{cases}$$

It has this corresponding matrix.

$$A = \left(\begin{array}{cccccc|c} 1 & 0 & -1 & 0 & 1 & 0 & 7 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 3 \end{array}\right)$$

The correspondence is obvious.

2.1 Elementary Row Operations

Row operations have the restriction that they must not alter the set of solutions for the corresponding system of equations. There are three.

1. Any two rows can be exchanged. For example, swapping rows 1 and 2 in A gives the following.

$$\left(\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 0 & 1 & 0 & 3 \end{array}\right)$$

2. Any row can be multiplied by a non-zero scalar. For example, multiplying row 3 in A by 5 gives the following.

$$\left(\begin{array}{cccccc|c} 1 & 0 & -1 & 0 & 1 & 0 & 7 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 5 & 0 & 0 & 0 & 5 & 0 & 15 \end{array}\right)$$

3. Any row can be multilplied by a non-zero scalar and added to another row. For example, multiplying row 1 by -2 and adding it to row 2 gives the following.

$$\left(\begin{array}{cccccc|c} 1 & 0 & -1 & 0 & 1 & 0 & 7 \\ -2 & 1 & 2 & 0 & -2 & -1 & -13 \\ 1 & 0 & 0 & 0 & 1 & 0 & 3 \end{array}\right)$$

2.2 Row Echelon Form and Solving Equation Matricies

First a theorem that should be obvious (we said it already). Then, an algorithm for solving linear systems of equations in matrix form.

Theorem Row operations do not change the solutions to the corresponding system of equations.

Gaussian Elimination Method of solving systems of equations. The first element of the first row must not be zero. Scale the first row so that its first element is one; the first non-zero element of each row is called the *pivot*. Using the first set the first element of all the other rows to zero. Repeat this process for the second element of the second row until the matrix is in *row echelon form*.

Row Echelon Form A matrix is in row echelon form when all non-zero rows are above any rows of all zeroes, the pivot is always to the right of the pivot above it, and the pivot is always one. (Short theorem: every matrix has a row echelon form.)

Example Suppose we want to solve the system in the following matrix.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & -1 & 1 & 2 \\ 1 & 0 & 1 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -5 & -5 & 0 \\ 0 & -2 & -2 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

Notice the third row states an impossibility, $0 = 2$. This system of equations is *inconsistent* and has no solutions. For the sake of example, suppose our matrix is now the following.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

We now can easily use back substitution to solve the system.

$$\begin{aligned} x_3 &= 2 \\ x_2 &= -x_3 = -2 \\ x_1 &= 1 - 2x_2 - 3x_3 = 1 - 2(-2) - 3(2) = -1 \end{aligned}$$

Or, written as the following triple.

$$\left\{ \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} \right\}$$

Reduced Row Echelon Form A matrix is in reduced row echelon form when in row echelon form and all columns are zero except the pivot. This form is convenient since the solutions to the corresponding system of equations in readily apparent in the far right column.

Row Echelon Form Remarks Suppose we have the following system.

$$\left(\begin{array}{ccccc|c} 1 & 3 & 2 & 5 & 2 & 1 \\ 0 & 1 & 3 & 7 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

All variables with pivotal columns are *lead variables.* All others are *free variables.*

$$\begin{array}{ccccccc} x_1 & +3x_2 & +5x_4 & +2x_5 & = & 1 & -2x_3 \\ & x_2 & +7x_4 & +x_5 & = & 2 & -3x_3 \\ & & x_4 & +2x_5 & = & 3 & \\ & & & x_5 & = & 4 & \end{array}$$

The corresponding system forms a set of solutions that are a *one-parameter family.*

A system in row echelon form is inconsistent if one of the rows has the form of all zeros except for the augmented column (which is equivalent to stating that $0 =$ some non-zero number.

If a system is in row echelon form and it is consistent then the set of solutions is a K-parameter family where K is the number of free variables.

2.3 Homogeneous Systems of Equations

A *homogeneous system of linear equations* has all constant terms of zero. It has the matrix form $A\vec{x} = \vec{0}$ and is not written with an augmented matrix form (the zero column is not useful).

Corollary Let $A\vec{x} = 0$ be a system with number of equations, m, number of variables, n. Then, if $n > m$, the number of free variables is greater than $n - m \geq 1$.

3 Matrix Algebra (Lecture 4)

3.1 Vector Notation

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

represents an n-dimensional vector (an element of $\Re^n$).

3.2 Vector Operations

- Addition

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \Longrightarrow \vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

  Note that the result remains in $\Re^n$.

- Multiplication by scalar Let $\alpha \in \Re$.

$$\alpha\vec{x} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

  Note, again, that the result remains in $\Re^n$.

3.3 Matrix Notation

An $m \times n$ matrix looks like the following.

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

For brevity, a matrix may also be written in the form $A = (a_{ij})$. In this class, we will also may make use of MATLAB notations such as the following.

$$\begin{pmatrix} A(1,:) \\ A(2,:) \\ \vdots \\ A(m,:) \end{pmatrix}$$

3.4 Matrix Operations

3.4.1 Addition

$$A = \begin{pmatrix} A(1,:) \\ A(2,:) \\ \vdots \\ A(m,:) \end{pmatrix}, \quad B = \begin{pmatrix} B(1,:) \\ B(2,:) \\ \vdots \\ B(m,:) \end{pmatrix}, \quad \implies A + B = \begin{pmatrix} A(1,:) + B(1,:) \\ A(2,:) + B(2,:) \\ \vdots \\ A(m,:) + B(n,:) \end{pmatrix}$$

Can also be noted as $A + B = ((a_{ij} + b_{ij})_{ij})$. An example.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix},$$

$$A + B = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 6 & 5 \end{pmatrix}$$

3.4.2 Multiplication by scalar

$$A = \begin{pmatrix} A(1,:) \\ A(2,:) \\ \vdots \\ A(m,:) \end{pmatrix}, \quad \alpha \in \Re \implies \alpha A = \begin{pmatrix} \alpha A(1,:) \\ \alpha A(2,:) \\ \vdots \\ \alpha A(m,:) \end{pmatrix}$$

Can also be noted as $\alpha A = ((\alpha a_{ij})_{ij})$. In example.

$$2A = \begin{pmatrix} 2 & 4 & 6 \\ 7 & 10 & 12 \end{pmatrix}$$

3.4.3 Multiplication by vector

Recall that the dot product of two vectors is as follows.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Hence, the dot product proceeds $\Re^n \times \Re^n \implies \Re$. One can similarly multiply a matrix by a vector.

$$A = \begin{pmatrix} A(1,:) \\ A(2,:) \\ \vdots \\ A(m,:) \end{pmatrix}$$

$$A \cdot \vec{x} = \begin{pmatrix} A(1,:) \cdot \vec{x} \\ A(2,:) \cdot \vec{x} \\ \vdots \\ A(m,:) \cdot \vec{x} \end{pmatrix} \in \Re^{m \times n}$$

In example.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad \vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$A \cdot \vec{x} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot (-1) + 3 \cdot 1 \\ 4 \cdot 1 + 5 \cdot (-1) + 6 \cdot 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

3.4.4 Multiplication by matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1k} \\ b_{21} & b_{22} & \dots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nk} \end{pmatrix}$$

(Note: dimensions critical!)

$$A \cdot B = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} & a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1n}b_{n2} & \cdots & a_{11}b_{1k} + a_{12}b_{2k} + \cdots + a_{1n}b_{nk} \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2n}b_{n1} & a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2n}b_{n2} & \cdots & a_{21}b_{1k} + a_{22}b_{2k} + \cdots + a_{2n}b_{nk} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \cdots + a_{mn}b_{n1} & a_{m1}b_{12} + a_{m2}b_{22} + \cdots + a_{mn}b_{n2} & \cdots & a_{m1}b_{1k} + a_{m2}b_{2k} + \cdots + a_{mn}b_{nk} \end{pmatrix}$$

$$A \cdot B = ((A(i,:) \cdot B(:,j))_{ij}) = \left(\left(\sum_{k=1}^{n} a_{ik}b_{kj} \right)_{ij} \right)$$

An example.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 3 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 1 \cdot 1 + 2 \cdot (-1) + 3 \cdot 1 & 1 \cdot 2 + 2 \cdot 1 + 3 \cdot 3 \\ 4 \cdot 1 + 5 \cdot (-1) + 3 \cdot 1 & 4 \cdot 2 + 5 \cdot 1 + 6 \cdot 3 \end{pmatrix} = \begin{pmatrix} 2 & 13 \\ 5 & 31 \end{pmatrix}$$

In this case, $B \cdot A$ also exists (as a 3×3 matrix).

There are a few properties of matrix multiplication worth noting.

- Associativity - $(A \cdot B)C = A(B \cdot C)$
- Distributivity - $(A + B)C = A \cdot C + B \cdot C$
- Compatibility with scalar multiplication - $\alpha(A \cdot B) = (\alpha \cdot A)B = A(\alpha \cdot B), \quad \alpha \in \Re$

If A is a *square matrix*, then the following notation may be used.

$$A^k = \underbrace{A \cdot A \cdots A}_{\text{k times}}, \quad k \in \Re$$

In example.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$$

$$A^k = \begin{pmatrix} 2^{k-1} & 2^{k-1} \\ 2^{k-1} & 2^{k-1} \end{pmatrix}$$

4 Linear Combinations (Lecture 5)

Let $\vec{v}, \vec{u_1}, \ldots, \vec{u_k} \in \Re^n$. We say that $\vec{v}$ can be *represented* as a *linear combination* of $\vec{u_1}, \ldots, \vec{u_k}$ if $\exists c_1, \ldots, c_k \in \Re$ such that $\vec{v} = c_1\vec{u_1}, \ldots, c_k\vec{u_k}$. For example.

$$\vec{u_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{u_2} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Now for $c_1 = 1$ and $c_2 = -1$.

$$c_1\vec{u_1} + c_2\vec{u_2} = 1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + -1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

Thus, $\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$ is a a linear combination of $\vec{u_1}$ and $\vec{u_2}$.

Any vector can be represented as a linear combination of the following.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Also stated

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 1 & 1 & 0 & y \\ 1 & 0 & 0 & z \end{array}\right]$$

for all $x, y, z \in \Re$.

Applications In a certain town, 30% of married women get divorced every year and 20% of the single women get married each year. In the town, there are 8000 married women and 2000 single women. Assuming the total number of women remains constant, what will be the statistics of married / single women after 1 year, 2 years, and 200 years.

$$\begin{pmatrix} \text{married} \\ \text{single} \end{pmatrix} = \begin{pmatrix} 8000 \\ 2000 \end{pmatrix}$$

A represents the change after one year.

$$A = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}$$

$$1 \text{ year} = A \begin{pmatrix} 8000 \\ 2000 \end{pmatrix} = \begin{pmatrix} \text{married after 1 year} \\ \text{single after 1 year} \end{pmatrix}$$

$$2 \text{ years} = A^2 \begin{pmatrix} 8000 \\ 2000 \end{pmatrix} = \begin{pmatrix} \text{married after 2 years} \\ \text{single after 2 years} \end{pmatrix}$$

$$200 \text{ years} = A^{200} \begin{pmatrix} 8000 \\ 2000 \end{pmatrix} = \begin{pmatrix} \text{married after 200 years} \\ \text{single after 200 years} \end{pmatrix}$$

Theorem Suppose we have a system $A \cdot \vec{x} = \vec{b}$ of the form $x_1 A(:,1) + x_2 A(:,2) + \cdots + x_n A(:,n) = \vec{b}$. The system is consistent if and only if it is representable as a linear combination of columns of A.

5 Square Matrices (Lecture 6)

They are square, i.e. in $\Re^{m \times m}$.

There is a special square matrix called the *identity matrix* with the following form.

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad I_n(i,j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

This matrix has the following very important property.

$$\forall A \in \Re^{m \times m}, \quad AI_n = I_n A = A$$

Proof.

$$AI_n(i,j) = \sum_{k=1}^{n} A(i,k) I_n(k,j) = A(i,j) I_n(j,j) = A(i,j)$$

5.1 Standard Basis

All we were told is it looks like this.

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \Re^n, \; e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \Re^n, \; \ldots, \; e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \Re^n$$

The *standard basis* is the set of those, $\{e_1, e_2, \ldots, e_n\}$.

5.2 Inverse Matrix

Sometimes for a square matrix, say $A \in \Re^{n \times n}$, there is a matrix $B \in \Re^{n \times n}$ such that $AB = BA = I_n$. In example.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Invertibility Let $A \in \Re^{n \times n}$. We say that A is *invertible* if there exists $B \in \Re^{n \times n}$ such that $AB = BA = I_n$. Also, B is called the *inverse*, A^{-1}, of A.

Uniqueness If A is invertible then there is a *unique* inverse. Proof. Let B_1, B_2 satisfy $AB = BA = I_n$.

$$B_2 = I_n B_2 = B_1 A B_2 = B_1 I_n = B_1$$

Terminology An invertible matrix is called *non-singular*. A non-invertible matrix is called *singular*.

Products of Invertible Matricies A product of two invertible real matricies *is invertible*. Moreover, $(AB)^{-1} = A^{-1}B^{-1}$. Proof.

$$(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = (AI_n)^{-1} = AA^{-1} = I_n$$

In the same way, $BAA^{-1}B^{-1} = I_n$.

6 Matrix Transpose (Lecture 6)

Some spiel about why $mn = \underbrace{n + \cdots + n}_{m \text{ times}}$ omitted here.

For a matrix $A = (a_{ij}) \in R^{m \times n}$, then $A^{\mathrm{t}} = (b_{ij}) \in \Re^{n \times m}$, where $b_{ij} = a_{ji}$. In example.

- $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad A^{\mathrm{t}} = \begin{pmatrix} 1 & 4 \\ 3 & 5 \\ 3 & 6 \end{pmatrix}$
- $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B^{\mathrm{t}} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

Properties Properties dump.

- $(\alpha A + \beta B)^{\mathrm{t}} = \alpha A^{\mathrm{t}} + \beta B^{\mathrm{t}}$
- $(AB)^{\mathrm{t}} = B^{\mathrm{t}} A^{\mathrm{t}}$
- If A is an invertible square matrix, then A^{t} is invertible and $(A^{\mathrm{t}})^{-1} = (A^{-1})^{\mathrm{t}}$.

Symmetry A matrix A is *symmetric* if $A^{\mathrm{t}} = A$. A symmetric matrix is always square. Example of a symmetric matrix.

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & 10 \\ 1 & 10 & 3 \end{pmatrix}$$

Graph Adjacency Matricies The *adjacency matrix* of a graph with all undirected edges is always symmetric. Theorem. Let C be a graph. Let A be the adjacency matrix of C. The number of k-walks from vertex i to vertex j is given by $A^k[i, j]$.

7 Elementary Matricies (Lecture 7)

Elementary matricies perform elementary row operations on other matricies when multiplied.

Let $M \in \Re^{m\times m}$ be invertible. Consider the system $A\vec{x} = \vec{b}$, for $A \in \Re^{m\times n}$, $\vec{b} \in \Re^m$. Then the systems $A\vec{x} = \vec{b}$ and $MA\vec{x} = M\vec{b}$ are equivalent.

Gaussian Algorithm Choose invertible matricies $E_1, E_2, \ldots, E_k$ such that the system $E_kE_{k-1}\cdots E_1A\vec{x} = E_kE_{k-1}\cdots E_1\vec{b}$ is much simpler (has reduced row echelon form), then the original system $A\vec{x} = \vec{b}$, $E_1, E_2, \ldots, E_k$ are *elementary matricies.* (Totally missed something here. To be updated.)

Elementary Matricies Take I_m. Performing any elementary matrix row operation on I_m produces an *elementary matrix.* In example.

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \to \begin{pmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Thus, there are three types of elementary matricies, each corresponding to one of the three elementary row operations.

- Exchange of row i with row j

  $$P_{ij} = \begin{pmatrix}e_1 & \cdots & e_{i-1} & e_j & e_{i+1} & \cdots & e_{j-1} & e_i & e_{j+1}\cdots & e_n\end{pmatrix}$$

  In example.

  $$P_{12} = \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, P_{12} = \begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}$$

- Multiplication of row i by $\alpha \neq 0$

  $$E_i(\alpha) = \begin{pmatrix}e_1 & \cdots & e_{i-1} & \alpha e_i & e_{i+1} & \cdots & e_n\end{pmatrix}$$

  In example.

  $$E_2\left(\frac{1}{2}\right) = \begin{pmatrix}1 & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & 1\end{pmatrix}, E_3(\sqrt{2}) = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \sqrt{2}\end{pmatrix}$$

- Addition of row i, multiplied by β, to row j

  $$E_{ij}(\beta) = \begin{pmatrix}e_1 & e_2 & \cdots & e_{j-1} & \beta e_i + e_j & e_{j+1} & \cdots & e_n\end{pmatrix}$$

  In example.

  $$E_{32}(\beta) = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & \beta\\ 0 & 0 & 1\end{pmatrix}$$

Proposition (Invertibility) All elementary matricies are invertible. Moreover,

- $(P_(ij))^{-1} = P_{ij}$
- $(E_i(\alpha))^{-1} = E_i\left(\frac{1}{\alpha}\right)$
- $(E_{ij}(\beta))^{-1} = E_{ij}(-\beta)$

Proposition (Correspondence to Elementary Row Operations) Let $A \in \Re^{m\times n}$. Then

- $A \overset{R_j \to R_i}{\to} B_1$ then $B_1 = P_{ij}A$
- $A \overset{\alpha R_i \to R_i}{\to} B_2$ then $B_2 = E_i(\alpha)A$
- $A \overset{R_i + \beta R_j \to R_i}{\to} B_3$ then $B_3 = E_{ij}(\beta)A$

In example.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad P_{12}A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}$$

Theorem (Characterization of Non-Singular, Square Matricies) Let $A \in \Re^{n\times n}$. TFAE (whatever that means):

1. A is non-singular (invertible)
2. $\forall \vec{b} \in \Re^n$ the system $A\vec{x} = \vec{b}$ has a unique solution
3. the system $A\vec{x} = \vec{0}$ has only the trivial solution in its solution set
4. A is representable as a product of elementary matricies

Proof. We will prove that $1 \implies 2 \implies 3 \implies 4 \implies 1$.

- $1 \implies 2$. If $\vec{x}_0 \in \Re^n$ is a solution of $A\vec{x} = \vec{b}$, then $A\vec{x}_0 = \vec{b}$. Applying A^{-1}, $A^{-1}A\vec{x}_0 = A^{-1}\vec{b}$.
- $2 \implies 3$. Clearly (2) is some statement which holds for every $\vec{b} \in \Re^n$ (3) is the statement but only for $\vec{0} \in \Re^n$.
- $3 \implies 4$. Let's solve $A\vec{x} \in \Re^n$.

  $$A \to \text{some elementary row operations} \to B$$

  B now has reduced row echelon form. We know the following about a system $B\vec{x} = \vec{0}$.

  number of free variables = number of variables−number of pivotal columns

  We also know that $B\vec{x} = \vec{0}$ having only the trivial solution implies the number of free variables is zero, the number of pivotal columns is the number of variables, and finally $B = I_n$.

- 4 $\implies$ 1. If $A = E_1 \dots E_k$ where $E_1, \dots, E_k$ are elementary matricies then A is invertible, and moreover $A^{-1} = E_k^{-1} E_{k-1}^{-1} \cdots E_1^{-1}$.

Exercise Represent $A = \begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}$ as a product of elementary matricies.

$$\begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix} \overset{R_2 - 5R_1 \to R_2}{\longrightarrow} \begin{pmatrix} 1 & 2 \\ 0 & -4 \end{pmatrix} \overset{-\frac{1}{4}R_2 \to R_2}{\longrightarrow} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \overset{R_1 - 2R_2 \to R_1}{\longrightarrow} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix} \overset{E_{12}(-2)}{\longrightarrow} \begin{pmatrix} 1 & 2 \\ 0 & -4 \end{pmatrix} \overset{E_2(-\frac{1}{4})}{\longrightarrow} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \overset{E_{21}(-5)}{\longrightarrow} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix} A = I_2$$

$$A = (E_{21}(-5))^{-1} (E_2(-\frac{1}{4}))^{-1} (E_{12}(-2))^{-1}$$