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Ordinal Games and Generalized Nash and Stackelberg Solutions[1]

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Abstract. The traditional theory of cardinal games deals with problems where the players are able to assess the relative performance of their decisions (or controls) by evaluating a payoff (or utility function) that maps the decision space into the set of real numbers. In that theory, the objective of each player is to determine a decision that minimizes its payoff function taking into account the decisions of all other players. While that theory has been very useful in modeling simple problems in economics and engineering, it has not been able to address adequately problems in fields such as social and political sciences as well as a large segment of complex problems in economics and engineering. The main reason for this is the difficulty inherent in defining an adequate payoff function for each player in these types of problems.

In this paper, we develop a theory of games where, instead of a payoff function, the players are able to rank-order their decision choices against choices by the other players. Such a rank-ordering could be the result of personal subjective preferences derived from qualitative analysis, as is the case in many social or political science problems. In many complex engineering problems, a heuristic knowledge-based rank ordering of control choices in a finite control space can be viewed as a first step in the process of modeling large complex enterprises for which a mathematical description is usually extremely difficult, if not impossible, to obtain. In order to distinguish between these two types of games, we will refer to traditional payoff-based games as cardinal games and to these new types of rank ordering-based games as ordinal games.

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In the theory of ordinal games, rather than minimizing a payoff function, the objective of each player is to select a decision that has a certain rank (or degree of preference) taking into account the choices of all other players. In this paper, we will formulate a theory for ordinal games and develop solution concepts such as Nash and Stackelberg for these types of games. We also show that these solutions are general in nature and can be characterized, in terms of existence and uniqueness, with conditions that are more intuitive and much less restrictive than those of the traditional cardinal games. We will illustrate these concepts with numerous examples of deterministic matrix games. We feel that this new theory of ordinal games will be very useful to social and political scientists, economists, and engineers who deal with large complex systems that involve many human decision makers with often conflicting objectives.

Key Words. Ordinal optimization, ordinal games, nonzero-sum games, Nash solution, Stackelberg solution.

1. Introduction

Systems that involve more than one decision maker are often optimized using the theory of games. This theory, as initially developed by von Neumann and Morgenstern (Ref. 1) and later by Nash (Ref. 2), requires that each point in the decision space be mapped through a payoff function[4] into a real number representing the value of this point to the decision maker (or player). In order to illustrate this concept, let us consider the following simple matrix game shown in Fig. 1. The first player (P1) has three choices in his decision space $X = \{x_1, x_2, x_3\}$ and the second player (P2) has also three choices $Y = \{y_1, y_2, y_3\}$. For each pair of choices $\{x_i, y_j\}$ the corresponding entries $J_1(x_i, y_j)$ and $J_2(x_i, y_j)$ in the matrix of Fig. 1 represent the payoffs incurred by both players, respectively. The objective of P1 is to choose $x_i \in X$ that yields the smallest value of J_1, and the objective of P2 is to choose $y_j \in Y$ that yields the smallest value of J_2. Obviously, the final outcome for each player is determined not only by its own choice, but also by the choice made by any other player. Hence, in making a choice, each player has to take into account the action of the other player.

Clearly, the usefulness of game theory as described above, is limited to problems where the payoff functions can be expressed and mathematically

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[4] Other common terms used instead of payoff function include utility function, objective function, cost function, loss function, performance function, profit function, etc. Such a function will either be maximized or minimized, depending on its definition.

		Player 2 (P2)		
		y_1	y_2	y_3
Player 1 (P1)	x_1	6.1 , 4.2	9.0 , 4.2	6.1 , 2.9
	x_2	7.0 , 10.5	6.1 , 9.3	2.9 , 7.3
	x_3	9.0 , 8.3	5.2 , 5.4	2.1 , 8.3

Fig. 1. A simple matrix game.

defined for all players in the game. It is not difficult, however, to imagine many decision-making problems where the payoff functions cannot be determined easily. Typical examples exist in many problems in the fields of social and political sciences, mathematical psychology, macroeconomics, and engineering of large complex systems or enterprises. For example, problems that most individuals face in buying a house, planning a vacation, or resolving a conflict, cannot be formulated easily and solved using the existing theory of games. Even in military applications, where the theory of games has received considerable attention, planning an air operation in the presence of an intelligent adversary is extremely difficult, if not impossible, to formulate using this theory of games. In all of these problems, the main difficulty lies in our inability to formulate and model appropriate payoff functions for all players. On the other hand, in the majority of such problems, instead of payoffs, the players may have certain preferences that can be expressed easily as a rank-ordering of the various options available to them. Furthermore, these preferences may be completely subjective in nature reflecting certain biases or experiences that the players may have. As a simple example to illustrate this idea, consider the situation of two friends, John and Mary, planning a weekend vacation together. John likes to go to the mountains and Mary prefers the beach. They both like to be together, if possible, but they have strong preferences for the various options available to them. Let us assume that, of the nine possible options, John ranks his preferences as illustrated in the table (or matrix) of Fig. 2. Mary may have completely different preferences. Let us suppose that hers are rankedas shown in Fig. 3.

		Mary (P2)		
		Mountain	Beach	Home
John (P1)	Mountain	1	6	5
	Beach	8	4	9
	Home	7	2	3

Fig. 2. John's preferences.

Superimposing both preferences,[5] we get the matrix game shown in Fig. 4.

Note that, in contrast to the game of Fig. 1, the game of Fig. 4 is formulated using preferential rankings instead of payoff functions. The entries in this game are positive integers representing the order or ranking of the various options for each player. In order to differentiate such games from the traditional, cardinal games, we will refer to them as ordinal games. Note that, in the ordinal game of Fig. 4, all nine options are rank ordered for each player. For example, both going to the mountains is John's 1st preference but Mary's 7th preference, and both going to the beach is Mary's 1st preference but John's 4th preference; however, John staying at home while Mary going to the beach is his 2nd preference but her 6th and so on. These are purely subjective preferences that may have no associated payoff or value.

In many ways, ordinal games can be viewed as extensions of ordinal optimization problems in the same manner as cardinal games are extensions of cardinal optimization problems. Ordinal optimization is a means of finding good, better, or best designs from a set of ordered options, rather than using a formal cardinal process of calculating the payoff cost or value of each option. It is a simple and yet very effective method of optimizing systems as demonstrated by Ho et al. (Ref. 3) in optimizing discrete event dynamic systems (DEDS).

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[5] In this table, we use the number 1 to indicate 1st preference, 2 for 2nd preference, and so on. Since there are 9 options, the number 9 will correspond to the 9th (last) preference. In the case of two or more options being equally ranked, the last preference will be less than 9th. Note that the use of integer numbers to indicate the ordering of the preferences, as we do in this paper, is purely optional. Symbols such as a, b, c, . . . , or *, #, @, . . . , could have been used as well.

		Mary (P2)		
		Mountain	Beach	Home
John (P1)	Mountain	7	8	9
	Beach	2	1	3
	Home	4	6	5

Fig. 3. Mary's preferences.

		Mary (P2)		
		Mountain	Beach	Home
John (P1)	Mountain	1 , 7	6 , 8	5 , 9
	Beach	8 , 2	4 , 1	9 , 3
	Home	7 , 4	2 , 6	3 , 5

Fig. 4. John and Mary's ordinal game.

The use of ordinal methods instead of payoff functions in decision-making problems is a concept that has received considerable attention in the past 20 years or so. The analytic hierarchy process (AHP) developed by Saaty in 1980 (Refs. 4–5) is a very effective tool for optimizing complex multicriteria decision-making problems. The AHP requires the decision maker to consider judgments, possibly subjective, about the relative importance of each criterion and specify a preference for each decision option with respect to each criterion. The outcome of this process is a prioritized ranking, indicating an overall preference, of the various decision options available to the decision maker. Along the same lines, the theory of moves developed by Brams in 1994 (Ref. 6) deals with models for conflict resolution that involve successive unilateral actions by the decision makers. These models, which differ in several crucial ways from game models (Ref. 7), are treated using an ordinal methodology that avoids the use of utility functions. Other methods that rely on an ordinal approach for representing preferences for the decision makers include the graph model conflict resolution (GMCR) developed by Kilgour et al. (Refs. 8–10) and the drama theory developed by Howard (Ref. 11). This ordinal approach has yet to find its way in the traditional formulation of zero-sum and nonzero-sum games as described by Von Neumann and Morgenstern (Ref. 1) and Nash (Ref. 2). What is needed is a theory for ordinal games that is conceptually simple so that it can be used easily in dealing with complex real-world problems in a large segment of societal decision-making problems and yet mathematically robust so that its results are meaningful and consistent with those of its counterpart, the theory of cardinal games.

2. Cardinal to Ordinal Games

An interesting aspect of a theory of ordinal games is that, when developed, it should also be able to handle cardinal games as well. This is so because cardinal games can be transformed easily into ordinal games as we illusrate in this simple example. Let us consider again the cardinal game shown in Fig. 1. Clearly, it is possible to map the payoff values for each player into a ranking of preference for the particular option that that payoff represents. For P1, a ranking of the payoffs would be as shown in Fig. 5, and for P2 as in Fig. 6. Using these rankings instead of the payoffs, the

Payoff	2.1	2.9	5.2	6.1	7.0	9.0
Rank	1	2	3	4	5	6
	Best → Worst					

Fig. 5. Rank-ordering for P1.

Payoff	2.9	4.2	5.4	7.3	8.3	9.3	10.5
Rank	1	2	3	4	5	6	7
	Best → Worst						

Fig. 6. Rank-ordering or P2.

		Player 2 (P2)		
		y_1	y_2	y_3
Player 1 (P1)	x_1	4, 2	6, 2	4, 1
	x_2	5, 7	4, 6	2, 4
	x_3	6, 5	3, 3	1, 5

Fig. 7. Ordinal game equivalent of the cardinal game of Fig. 1.

matrix for the game can be reformulated as shown in Fig. 7. The entries in this new matrix are now the rankings of each pair of choices for P1 and P2, respectively. Thus, for example, the pair $\{x_1, y_1\}$ is ranked 4th best for P1 and 2nd best for P2, and so on. The highest ranked option or P1 is clearly $\{x_3, y_3\}$, and for P2 is $\{x_1, y_3\}$.

3. Ordinal Matrix Games: Formulation

Let $X = \{x_1, x_2, \ldots, x_n\}$ represent the decision space for P1, and let $Y = \{y_1, y_2, \ldots, y_m\}$ represent the decision space for P2. As mentioned earlier, cardinal games are based on the concept of payoffs. Let $J_1(x, y)$ and $J_2(x, y)$ for $x \in X$ and $y \in Y$ denote the $n \times m$ matrices of payoffs associated with the decision pair $\{x, y\} \in X \times Y$ for P1 and P2, respectively.

Definition 3.1. A two-player matrix cardinal game is defined as a pair of $n \times n$ matrices J_1 and J_2. The ijth entries in J_1 and J_2 represent the payoffs associated with the decision pair $\{x_i, y_j\}$ for P1 and P2, respectively.

Example 3.1. For the cardinal game of Fig. 1, we have

$$J_1 = \begin{bmatrix} 6.1 & 9.0 & 6.1 \\ 7.0 & 6.1 & 2.9 \\ 9.0 & 5.2 & 2.1 \end{bmatrix}, \qquad J_2 = \begin{bmatrix} 4.2 & 4.2 & 2.9 \\ 10.5 & 9.3 & 7.3 \\ 8.3 & 5.4 & 8.3 \end{bmatrix}.$$

Ordinal games, on the other hand, are based on the concept of preferential rank ordering. Let $R_1(x, y)$ and $R_2(x, y)$ for $x \in X$ and $y \in Y$ denote the $n \times m$

matrices of rank ordering of the decision pair $\{x, y\} \in X \times Y$ for P1 and P2, respectively. We note that the entries of $R_1(x, y)$ and $R_2(x, y)$ are integers,

$$R_1(x_i, y_j) = \{1, 2, \ldots, W_1\}, \qquad \text{for } i = 1, \ldots, n \text{ and } j = 1, \ldots, m, \tag{1}$$

$$R_2(x_i, y_j) = \{1, 2, \ldots, W_2\}, \qquad \text{for } i = 1, \ldots, n \text{ and } j = 1, \ldots, m, \tag{2}$$

where W_1 and W_2 represent the last ranked (worst) options for P1 and P2, respectively. Note that $W_i \leq nm$, for $i = 1, 2$, allowing for some decision pairs to have duplicate (same) rank. Thus, the notation $R_i(x_a, y_b) = k$ means that the decision pair $\{x_a, y_b\}$ is ranked as the kth best option for player i. It is possible for two decision pairs to be equally ranked for a player. This would be indicated by $R_i(x_a, y_b) = R_i(x_c, y_d)$, meaning that, for player i, the decision pair $\{x_a, y_b\}$ has the same preference (rank) as $\{x_c, y_d\}$.

Definition 3.2. A two-player matrix ordinal game is defined as a pair of $n \times m$ matrices R_1 and R_2. The ijth entries of R_1 and R_2 represents the preferential rankings of the decision pair $\{x_i, y_j\}$ for P1 and P2, respectively.

Example 3.2. For the ordinal game of Fig. 4 (from now on, we will refer to it as OG1), we have

$$R_1 = \begin{bmatrix} 1 & 6 & 5 \\ 8 & 4 & 9 \\ 7 & 2 & 3 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} 7 & 8 & 9 \\ 2 & 1 & 3 \\ 4 & 6 & 5 \end{bmatrix}.$$

Definition 3.3. Let M be an $n \times m$ matrix whose entries are real numbers. We will define an associated rank-ordered matrix M^{o} as the $n \times m$ matrix in which each number is replaced by its order (rank) in the set $\{M_{ij}$, for $i = 1, \ldots, n$ and $j = 1, \ldots, m\}$ starting with the smallest.

Note that, in the above definition, it is implied that entries in M that are equal are assigned the same rank.

Example 3.3. A few illustrative examples are given below.

If $v = [7.2, 26.2, 4.5, 8.6]$, then $v^{\text{o}} = [2, 4, 1, 3]$.

$$\text{If } u = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}, \qquad \text{then } u^{\text{o}} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

$$\text{If } J_1 = \begin{bmatrix} 6.1 & 9.0 & 6.1 \\ 7.0 & 6.1 & 2.9 \\ 9.0 & 5.2 & 2.1 \end{bmatrix}, \qquad \text{then } J_1^{\text{o}} = \begin{bmatrix} 4 & 6 & 4 \\ 5 & 4 & 2 \\ 6 & 3 & 1 \end{bmatrix}.$$

$$\text{If } J_2 = \begin{bmatrix} 4.2 & 4.2 & 2.9 \\ 10.5 & 9.3 & 7.3 \\ 8.3 & 5.4 & 8.3 \end{bmatrix}, \qquad \text{then } J_2^{\mathrm{o}} = \begin{bmatrix} 2 & 2 & 1 \\ 7 & 6 & 4 \\ 5 & 3 & 5 \end{bmatrix}.$$

As mentioned earlier, every cardinal game as an associated ordinal game.

Definition 3.4. Given a cardinal game defined by the matrices J_1 and J_2, the associated ordinal game is defined by the matrices J_1^{o} and J_2^{o}.

Example 3.4. The cardinal game of Fig. 1, whose matrices J_1 and J_2 are in Example 3.1, has an associated ordinal game defined by the matrices

$$J_1^{\mathrm{o}} = \begin{bmatrix} 4 & 6 & 4 \\ 5 & 4 & 2 \\ 6 & 3 & 1 \end{bmatrix}, \qquad J_2^{\mathrm{o}} = \begin{bmatrix} 2 & 2 & 1 \\ 7 & 6 & 4 \\ 5 & 3 & 5 \end{bmatrix},$$

as has been illustrated in Fig. 7.

It is important to note that a cardinal game has a unique associated ordinal game. However, for a given ordinal game, there are infinitely many (continuum) cardinal games whose associated ordinal game is the same as the given ordinal game. The use of rank ordering, instead of the actual values of the payoffs will allow for considerable variability in the payoffs without changing the ranking. That is, the continuous payoff space can be discretized into a (finite) countable number of levels. To illustrate this concept, let us assume that the game in Fig. 1 is reformulated so that the payoffs for a given pair of choices have ranges of value (instead of specific values), as illustrated in Fig. 8. All the cardinal games that are formulated with the

		Player 2 (P2)		
		y_1	y_2	y_3
Player 1 (P1)	x_1	(5.5 - 6.5] , (3.5 -4.5]	(8 - ∞) , (3.5 - 4.5]	(5.5 - 6.5] , [0 - 3.5]
	x_2	(6.5 - 8] , (9.5 - ∞)	(5.5 - 6.5] , (8.5 - 9.5]	(2.5 - 4] , (6 - 7.5]
	x_3	(8 - ∞) , (7.5 - 8.5]	(4 - 5.5] , (4.5 - 6]	[0 - 2.5] , (7.5 - 8.5]

Fig. 8. Ranges of payoffs in the cardinal game of Fig. 1 that will yield the same associated ordinal game shown in Fig. 7.

values of payoffs within these ranges are associated with the same ordinal game defined by J_1^{o} and J_2^{o} as in Example 3.4 above. Essentially, this says that, when the payoffs are being determined or calculated to form a cardinal game, the players have considerable latitude in how accurately their calculations need to be performed. This lack of sensitivity of the formulation of the game, and eventually its solution, to variations in the payoff functions is a very important advantage that ordinal games have over cardinal games. In other words, a game formulated in the ordinal sense is far more robust than a game formulated in the cardinal sense.

4. Generalized Nash Solutions for Ordinal Games

The Nash solution represents an equilibrium point when each player reacts to the other player by choosing the option that gives him the smallest payoff. The definition of a Nash solution (Ref. 2) for cardinal games is well known and is given below for the sake of completeness.

Definition 4.1. A Nash solution for a cardinal game is a pair of choices $\{x^N, y^N\}$ that satisfies the following inequalities (assuming that the players wish to minimize costs; if the J's are payoff functions to be maximized, the inequalities are reversed):

$$J_1(x^N, y^N) \leq J_1(x, y^N), \qquad \forall x \in X, \tag{3}$$

$$J_2(x^N, y^N) \leq J_2(x^N, y), \qquad \forall y \in Y. \tag{4}$$

Note that the first inequality (3) guarantees P2 that, by choosing y^N, P1 has no better choice than x^N; and the second inequality (4) guarantees P1 that, by choosing x^N, P2 has no better choice than y^N.

Example 4.1. In the cardinal game of Fig. 1, $\{x_3, y_2\}$ is the only pair of options that satisfies inequalities (3) and (4). It is the only Nash solution for the game.

In order to be able to define a concept similar to the Nash solution for ordinal games, we need first the following definition.

Definition 4.2. Let M be an $n \times m$ matrix whose entries are real numbers. We will define a column rank-ordered matrix M^{co} as the corresponding $n \times m$ matrix in which each column vector is replaced by its corresponding ordered column. That is,

$$M^{\mathrm{co}} = [m_{\mathrm{c}1}^{\mathrm{o}} \vdots m_{\mathrm{c}2}^{\mathrm{o}} \vdots \cdots \vdots m_{\mathrm{cm}}^{\mathrm{o}}]. \tag{5}$$

Similarly, we define a row rank-ordered matrix M^{ro} as the corresponding $n \times m$ matrix in which each row vector is replaced by its corresponding ordered row. That is,

$$M^{\mathrm{ro}} = \begin{bmatrix} m^{\mathrm{o}}_{\mathrm{r}1} \\ \cdots \\ m^{\mathrm{o}}_{\mathrm{r}2} \\ \cdots \\ \vdots \\ \cdots \\ m^{\mathrm{o}}_{\mathrm{r}n} \end{bmatrix} \tag{6}$$

Example 4.2. Given

$$M = \begin{bmatrix} 4 & 7 & 1 \\ 5 & 3 & 4 \\ 6 & 2 & 6 \end{bmatrix},$$

then

$$M^{\mathrm{co}} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 2 \\ 3 & 1 & 3 \end{bmatrix}, \qquad M^{\mathrm{ro}} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix}.$$

Definition 4.3. Given an ordinal game defined by the matrices R_1 and R_2. Each pair of decisions $\{x_i, y_j\}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ is defined as a Generalized Nash (GN) solution of order $\{R^{\mathrm{co}}_1(x_i, y_j), R^{\mathrm{ro}}_2(x_i, y_j)\}$.

Example 4.3. For OG1 (see Example 3.2), we have

$$R^{\mathrm{co}}_1 = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix}, \qquad R^{\mathrm{ro}}_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix}.$$

Hence, $\{x_1, y_1\}$ is a GN solution of order $\{1, 1\}$ and $\{x_1, y_2\}$ is a GN solution of order $\{3, 2\}$, etc. A GN solution of order $\{3, 2\}$ is a Nash equilibrium point when P1 reacts to choices of P2 by choosing its 3rd ranked option while P2 reacts to choices of P1 by choosing its 2nd ranked option. Similarly, if both players react always by choosing their 2nd preference choice, then the GN equilibrium will be of order $\{2, 2\}$. A GN of order $\{2, 2\}$ does not exist in this example. On the other hand, a GN solution of order $\{3, 3\}$ does exist. It corresponds to $\{x_2, y_3\}$ and represents a GN equilibrium point when each player reacts by choosing its 3rd (i.e., worst) option. From a practical point of view, this would be the least desirable or worst ranked Nash equilibrium for the game.

The above definition says essentially that, in an ordinal game, every pair of choices is a GN solution of a certain order. A generalized Nash solution of order $\{R_1^{\mathrm{co}}(x_i, y_j), R_2^{\mathrm{Ro}}(x_i, y_j)\}$ represents a Nash equilibrium point when P1 reacts by choosing its $R_1^{\mathrm{co}}(x_i, y_j)$th preference option and P2 reacts by choosing its $R_2^{\mathrm{Ro}}(x_i, y_j)$th preference option. Clearly, this concept of generalized Nash solutions for ordinal games is much richer than the corresponding concept in cardinal games.

Definition 4.4. For an ordinal game, if there exists a generalized Nash (GN) solution of order $\{1, 1\}$, then we shall call this Nash solution an optimal Nash (ON) solution for the game.

Example 4.4. For OG1 (see Example 4.3), the pair $\{x_1, y_1\}$ is an optimal Nash solution for the game. This corresponds to the most desirable or highest ranked Nash equilibrium for the game. Similarly, for the OG2 of Fig. 7, we have

$$R_1^{\mathrm{co}} = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 2 & 2 \\ 3 & 1 & 1 \end{bmatrix}, \qquad R_2^{\mathrm{Ro}} = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix},$$

and the optimal Nash solution is $\{x_3, y_2\}$.

The concept of an optimal Nash solution for ordinal games is equivalent to the standard concept of Nash solution in cardinal games. However, as is well known, Nash solutions in pure strategies do not always exist in cardinal games. In ordinal games, the concept of generalized Nash solutions provides for numerous alternative Nash solutions in case the optimal Nash solution does not exist.

5. Generalized Stackelberg Solutions for Ordinal Games

Another useful solution concept for cardinal games is when one player has the capability of announcing its choice before the other player. In the example of Fig. 1, let us assume that P2 has such a capability. P2 would then figure out how P1 will react to its three possible options, and choose the option that is most favorable for itself. Thus, if P2 chooses y_1, then P1 would obviously react by choosing x_1, since this choice would produce the least value of the payoff. Similarly, if P2 chooses y_2, then P1 would react by choosing x_3, and finally if P2 chooses y_3, then P1 would react by also choosing x_3. Among these three possible choices, clearly the best choice for P2 is y_1, resulting in a choice or reaction of x_1 by P1, and payoffs of 6.1 and 4.2 for P1 and P2, respectively. In the terminology of cardinal games, this solution is referred to as the Stackelberg solution with P2 as leader and P1 as follower (Refs. 12–14). Clearly, this solution is completely different from the cardinal Nash solution and is predicated on different nonsymmetric roles assigned to the two players. The formal definition of the cardinal Stackelberg solution with P2 as leader[6] is given below.

Definition 5.1. Let $D_1 = \{d_1(y_1), d_1(y_2), \ldots, d_1(y_m)\}$ denote the set of reactions of P1 that minimize J_1 against all possible choices $\{y_1, y_2, \ldots, y_m\}$ available to P2; i.e.,

$$J_1(d_1(y), y) \leq J_1(x, y), \qquad \forall y \in Y, x \in X. \tag{7}$$

Then, the cardinal Stackelberg solution with P2 as leader and P1 as follower is a pair of choices $\{x^{S_2}, y^{S_2}\}$, where $x^{S_2} = d_1(y^{S_2}\}$ that satisfies the inequality

$$J_2(d_1(y^{S_2}), y^{S_2})) \leq J_2(d_1(y), y), \qquad \forall y \in Y. \tag{8}$$

Thus, the cardinal Stackelberg solution with P2 as leader represents the best pair $\{d_1(y_j), y_j\}$ in $D_1 \times Y$ that P2 can choose to minimize J_2. It is interesting to note that, by comparing the inequalities (8) and (4), it is easy to see that the cardinal Stackelberg solution will always yield a better (or at least an equal) payoff for the leader than the cardinal Nash solution. In order to be able to define a concept of a Stackelberg solution for ordinal games, we need first the following definition.

Definition 5.2. Let M be an $n \times m$ matrix of real numbers, and let Q^{co} be an $n \times m$ column rank-ordered matrix that corresponds to a matrix Q (see Definition 4.2). We will define a Q^{co} rank-ordered matrix $M^{Q^{\text{co}}}$ as the corresponding $n \times m$ matrix obtained by replacing each entry in M by its corresponding rank in the set of elements that are similarly ranked by the matrix Q^{co}. That is,

$$M^{Q^{\text{co}}}_{\{I_k\}} = M^{\text{o}}_{\{I_k\}}, \qquad \forall I_k = \{i, j \text{ such that } Q^{\text{co}}_{i,j} = k,$$
$$\text{for } k = 1, 2, \ldots, m\}. \tag{9}$$

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[6] The definition of the Stackelberg solution with P1 as leader and P2 as follower is obtained by interchanging the subscripts 1 and 2 in the given definition with P2 as leader.

Example 5.1. Consider the 3×2 matrix

$$M = \begin{bmatrix} 8.1 & 9.1 \\ 4.5 & 2.0 \\ 7.8 & 3.6 \end{bmatrix}$$

and the column rank-ordered matrix

$$Q^{\mathrm{co}} = \begin{bmatrix} 3 & 3 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Then,

$$M^{Q^{\mathrm{co}}} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 1 \end{bmatrix}.$$

That is, the entries in M that are ranked 1 by the matrix Q^{co}, i.e.,

$$M_{\{I_1\}} = [2.0, 7.8],$$

are now ranked and replaced by their own ranking in this vector, i.e.,

$$M^{\mathrm{o}}_{\{I_1\}} = [1, 2].$$

Similarly, the entries in M that are ranked 2 by the matrix Q^{co}, i.e.,

$$M_{\{I_2\}} = [4.5, 3.6],$$

are now ranked and replaced by their own ranking, i.e.,

$$M^{\mathrm{o}}_{\{I_2\}} = [2, 1],$$

and so on.

Example 5.2. Consider the 3×3 matrix

$$R_2 = \begin{bmatrix} 7 & 8 & 9 \\ 2 & 1 & 3 \\ 4 & 6 & 5 \end{bmatrix}$$

and the column rank-ordered matrix

$$R_1^{\mathrm{co}} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix}$$

of OG1 (see Examples 3.2 and 4.3). Then,

$$R_2^{R_1^{\mathrm{co}}} = \begin{bmatrix} 3 & 3 & 3 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}.$$

Here, the entries in R_2 that are ranked 1 by the matrix R_1^{co}, i.e.,

$$M_{\{I_1\}} = [7, 6, 5],$$

are now ranked and replaced by their rankings, i.e.,

$$M^{\mathrm{o}}_{\{I_1\}} = [3, 2, 1],$$

and so on.

Definition 5.3. Given an ordinal game defined by the matrices R_1 and R_2. Each pair of decisions $\{x_i, y_j\}$, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, is defined as the generalized Stackelberg (GS2) solution of order $\{R_1^{\mathrm{co}}(x_i, y_j), R_2^{R_1^{\mathrm{co}}}(x_i, y_j)\}$ with P2 as leader and P1 as follower.

Example 5.3. For OG1 (see Examples 3.2, 4.3, 5.2), we have

$$R_1^{\mathrm{co}} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix}, \qquad R_2^{R_1^{\mathrm{co}}} = \begin{bmatrix} 3 & 3 & 3 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}.$$

Hence, $\{x_1, y_1\}$ is a GS2 solution of order $\{1, 3\}$ and $\{x_2, y_3\}$ is a GS2 solution of order $\{3, 2\}$, and so on. A GS2 solution of order $\{3, 2\}$ is a Stackelberg solution when P1 reacts to choices of P2 by always choosing his 3rd ranked option and when P2, being the leader, always chooses his 2nd ranked option. By the same reasoning, if both players always choose their 2nd preference, then the GS2 solution will be of order $\{2, 2\}$. The GS2 of order $\{2, 2\}$ in this game is $\{x_3, y_1\}$. Note that, as pointed out in Example 4.3, a GN solution of order $\{2, 2\}$ does not exist for this game.

The above definition essentially says that, in an ordinal game, every pair of choices is a GS2 solution of a certain order. The GS2 solution of order $\{R_1^{\mathrm{co}}(x_i, y_j), R_2^{R_1^{\mathrm{co}}}(x_i, y_j)\}$ represents a Stackelberg solution when P1 reacts by choosing his $R_1^{\mathrm{co}}(x_i, y_j)$th preference option and P2, being the leader, always chooses his $R_2^{R_1^{\mathrm{co}}}(x_i, y_j)$th preference option. Clearly, as in the case of the Nash concept, this concept of generalized Stackelberg solutions for ordinal games is much richer than the corresponding concept in cardinal games.

An interesting observation is that, while the generalized Nash solutions may not always exist for all possible orders, it can be shown that indeed, provided there are no duplicate rankings, all generalized Stackelberg solutions (GS1 and GS2) do exist always for all possible orders. This is stated in the following theorem.

Theorem 5.1. Given an ordinal game with no duplicate rankings, all GS2 and GS1 solutions of order $\{i,j\}$, for $i=1,\ldots,n$ and $j=1,\ldots,m$, exist and are unique.

Proof. We will prove the theorem for GS2 only. The proof for GS1 is almost identical. In forming the column rank-ordered matrix $R_1^{\mathrm{co}}(x_i, y_j)$ for the follower (P1), it is clear that every column will contain all the integers $\{1, 2, \ldots, n\}$. Now, $R_2^{R_1^{\mathrm{co}}}(x_i, y_j)$ is a matrix that has been rank-ordered according to $R_1^{\mathrm{co}}(x_i, y_j)_j$ and consequently itself will contain all the integers $\{1, 2, \ldots, m\}$. Hence, every pair of integers in the set $\{i,j$ for $i=1,\ldots,n$ and $j=1,\ldots,m\}$ will be included in the pair of matrices $\{R_1^{\mathrm{co}}(x_i, y_j), R_2^{R_1^{\mathrm{co}}}(x_i, y_j)\}$. That is, GS2 solutions of all orders will exist always. ☐

Finally, the concept of an optimal Stackelberg solution corresponds to the case where both players always choose based on their first preference. This is summarized in the following definition.

Definition 5.4. For an ordinal game, the generalized Stackelerg solution (GS2) of order $\{1, 1\}$ is called the optimal Stackelberg solution for P2 and the generalized Stackelberg solution (GS1) of order $\{1, 1\}$ is called the optimal Stackelberg solution for P1.

Example 5.4. For OG1 (see Example 5.3), the optimal Stackelberg solution for P2 is $\{x_3, y_3\}$. This is the only pair of choices that has an order of $\{1, 1\}$.

Note that the concept of optimal Stackelberg solution for ordinal games is equivalent to the standard concept of Stackelberg solution in cardinal games.

6. Conclusions

In this paper, we developed a new theory of games where, instead of calculating payoffs, the players are able to rank order their decision choices against decision choices of the other players. We labeled these types of games as ordinal games. We developed the concepts of generalized Nash and Stackelberg solutions for ordinal games and showed that these are much richer concepts than their counterparts in traditional cardinal games. We feel that this new theory of ordinal games will provide a very useful alternative to many social and political scientists, economists, and engineers who deal with decision-making problems that cannot be formulated using payoff functions. We also feel that the theory of ordinal games can be a very valuable supplement to the theory of cardinal games in dealing with problems that can be formulated using the traditional payoff functions approach.

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