Towards a Compact and Computer-Adapted Formulation of the Dynamics and Stability of Multi Rigid Body Systems

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Abstract—**The dynamics of rigid bodies coupled by holonomic and non-holonomic constraints are formulated by the Newton - Euler method - employing a compact notation. The compact notation involves the use of two three by three matrices A and B and the totality of constraint vector C. The Lagrangian and Newton - Euler methods are related for a one - link rigid body in order to introduce the methodology of the paper in full detail. Stability and control of the resulting nonlinear systems are investigated by the use of Lyapunov methods. Digital computer simulations of typical movements are carried out in order to demonstrate feasibility of the formulation and the approach.**

Index Terms—**Rigid body systems, Lagrangian and Newton - Euler formulations, compact and computer-adapted dynamics, Lyapunov stability.**

I. Introduction

Systems of connected rigid bodies with holonomic, nonholonomic and soft constraints appear frequently in many current applications [1], [2], [3]. Specific examples and details can be found in references [4], [5], [6]. Rigid body systems can be used to model robotic, humanoid and locomotion systems. Compact representations of the equations of motion facilitate computer studies of stability, control and computer simulation. It is paramount to rely on tensors, dyads, matrices, and vectors rather than on scaler variables in such applications due to the large dimensionality of the systems involved.

This paper is concerned with the introduction of a compact formulation for the above objectives. The method makes use of three matrices: A, B and C. Matrix A describes the transformation of vectors from the body coordinate system (bcs) to the inertial coordinate system (ics). Matrix B relates the angular velocity and the derivatives with respect to time of Euler angles. Matrix C describes the constraints of interaction with, connection to and contact with other objects and with the environment. In order to demonstrate the properties of the above matrices, and make them more familiar, the Lagrangian and Newton-Euler formulation are developed for one rigid body, and the inherent relations between the two are exposed.

The formulation is modular, and yields itself to standard Lyapunov stability methods. A procedure is developed for global stability of such systems in the range of system states and parameters where the Lipschitz condition is satisfied. The procedure outlined here also allows the construction of Lyapunov functions in a systematic way, analogous to the methods of construction of such functions for linear stable time-invariant systems [7]. The two rigid body system is considered in reference [8].

Stability is achieved by the introduction of certain physically-based feedback structures and the use of the total energy of the system as a Lyapunov function [9], [10], [11]. The starting point is a suitable state space of translation in an inertial coordinate system (ics) where Cartesian positions and velocities constitute six states; and a suitable state space of rotation where Euler angles, and the angular velocity of the body in a body-based orthogonal coordinate system (bcs) are the other six states [12], [13].

II. Single Rigid body

Let Θ and Ω be, respectively, the Euler angles in the ics and the angular velocity vector of the body expressed in the bcs. Let X and V be the translational vectors of position and velocity of the center of gravity of the body in the ics system. Let Λ be a 3-vector of force acting on a point of the body whose coordinates are vector R in bcs. In connection with vectors R and Ω, we define the skew symmetric 3×3 matrices $\breve{\mathcal{R}}$ and $\breve{\Omega}$. Let vector R have components r_1, r_2, and r_3. The skew symmetric matrix $\breve{\mathcal{R}}$ is defined as:

$$\breve{\mathcal{R}} = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}.$$

Let vectors G be the gravity vector expressed in the ics. Let A be the orthonormal transformation of vectors from the bcs to ics. The inverse transformation is A'. The Newton - Euler equations of motion are [12]:

$$\begin{aligned} \dot{X} &= V \\ M\dot{V} &= G + \Lambda \\ \dot{\Theta} &= B(\Theta)\Omega \\ J\dot{\Omega} &= f(\Omega) + N + \breve{\mathcal{R}}A'\Lambda \end{aligned} \tag{1}$$

in the above formulation

$$f = -\breve{\Omega}J\Omega,$$

and

$$M = m \times I$$

where I is the 3×3 identity matrix. The 3×3 matrix J is the positive definite symmetric moment of inertia matrix in the body coordinate system. The matrices $A(\Theta)$ and $B(\Theta)$ that appear above are given below. Matrix A depends on the choice of Euler angles. A roll, pitch, yaw sequence is adopted here.

Let $A_1(\theta_1)$, $A_2(\theta_2)$, and $A_3(\theta_3)$ be defined by

$$A_1(\theta_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}.$$

$$A_2(\theta_2) = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}.$$

$$A_3(\theta_3) = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now $A(\Theta)$ can be defined:

$$A(\Theta) = A_1(\theta_1)A_2(\theta_2)A_3(\theta_3).$$

The matrix $B(\Theta)$ is given by:

$$B(\Theta) = \begin{bmatrix} \frac{cos\theta_3}{cos\theta_2} & \frac{-sin\theta_3}{cos\theta_2} & 0 \\ sin\theta_3 & cos\theta_3 & 0 \\ \frac{-sin\theta_2 cos\theta_3}{cos\theta_2} & \frac{sin\theta_2 sin\theta_3}{cos\theta_2} & 1 \end{bmatrix}.$$

The inverse of matrix B is:

$$B^{-1}(\Theta) = \begin{bmatrix} \cos\theta_3 cos\theta_2 & sin\theta_3 & 0 \\ -sin\theta_3 cos\theta_2 & cos\theta_3 & 0 \\ \sin\theta_2 & 0 & 1 \end{bmatrix}.$$

Five properties of these matrices, important for the later derivations in this paper are listed here. The detailed derivation of these properties is left out. The first two properties involve matrix A. The first one is:

$$\dot{A} = A\breve{\Omega} \qquad (2)$$

The second property is:

$$(\breve{A\Omega}) = A\breve{\Omega}A' \qquad (3)$$

The inverse of B satisfies the following equation:

$$\dot{B^{-1}} + \breve{\Omega}B^{-1} = \partial(B^{-1}\dot{\Theta})/\partial\Theta \qquad (4)$$

This equation is important in linking the Lagrangian and the Newton - Euler formulations. The fourth property involves both A and B: Let R be a constant vector. It can be proven that

$$\partial AR/\partial\Theta = -A(\breve{R})B^{-1} \qquad (5)$$

The fifth property, needed in reduction of the equations of motion for a holonomic connection constraint, is the following identity:

$$\breve{\Omega}\breve{R}\breve{R}\Omega = \breve{R}\breve{\Omega}\breve{R}\Omega \qquad (6)$$

A. Constrained Holonomic Motion

For ease of expression let Z be the 12-vector of X,V, Θ and Ω. Let Φ be the 6-vector of Θ and Ω. Suppose an arbitrary point of the body with coordinate vector R , expressed in the bcs, is fixed to the origin of the ics. The system is represented by equation 1 where Λ is the three-vector of the forces of constraint. The holonomic constraint is described symbolically and exactly as follows. $C(Z) = 0$ or

$$X + AR = 0.$$

The motion sub manifold, on which the motion takes place, can be described parametrically as follows:

$$\begin{bmatrix} X \\ V \\ \Theta \\ \Omega \end{bmatrix} = \begin{bmatrix} -AR \\ A(\breve{\Omega})R \\ \Theta \\ \Omega \end{bmatrix} \qquad (7)$$

Diffrentiation equation 7 with respect to time, and separating the result along the $\dot{\Theta}$ and $\dot{\Omega}$ gives

$$\dot{Z} = \begin{bmatrix} A\breve{R}\Omega \\ A\breve{\Omega}\breve{R}\Omega \\ \dot{\Theta} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ A\breve{R} \\ 0 \\ I \end{bmatrix} \dot{\Omega} \qquad (8)$$

In order to project the motion onto the Φ space, both sides of equation 1 are multiplied by the transpose of the right most column of equation 8. Since the vector involving Λ in equation 1 is orthogonal to the sub manifold of motion, and the vector column of equation 8 is tangent to the sub manifold of motion, the constraint force Λ is eliminated from the resulting equations [4]. Define

$$J1 = J - (\breve{R})M(\breve{R}).$$

This matrix is the inertia matrix of the body relative to a coordinate system centered at position R on the body, and parallel with the previously defined bcs. The bcs here may or may not lie along the body's principal axes. The reduced equations of motion, after some manipulation, are:

$$\dot{\Theta} = B(\Theta)\Omega$$
$$J1\dot{\Omega} = \breve{\Omega}J1(\Omega) + N - \breve{R}A'G \qquad (9)$$

We note that the body coordinate system is still the same, but the matrix J is modified to $J1$. However, the general form of the rotational equation remains the same as in equation 1.

B. Further Constraints

Let us assume that the above rigid body, attached to the ics at one point, is further constrained by holonomic or non-holonomic constraints. The case of holonomic constraints is discussed in [10]. Non-holonomic constraints that limit the angular velocity of the body Ω are considered here.

B.1 Two Constraints

Let us assume the rigid body rotates about a fixed given axis. Let this vector be described, in the ics, by the unit vector τ, and, in the bcs, by the unit vector h. If, at the initial time, the two platforms of the ics and bcs are aligned, these two unit vectors are the same. Let w be a function of time, or, for the moment, a proportionality constant. The angular velocity of the body, in the ics, is described by

$$A\Omega = \tau w \tag{10}$$

Let C' be a 2×3 matrix whose rows are orthogonal to vector τ The constraint above can be written in the standard form

$$C'A\Omega = 0.$$

The latter can be written in terms of Θ and $\dot{\Theta}$ for inclusion in the Lagrangian equations of motion:

$$(C'A(B^{-1}))\dot{\Theta} = 0 \tag{11}$$

Let the two torques of constraint, expressed in the bcs, be M_c.

The contribution of the constraint torques to the dynamics[14] is

$$(B^{-1})'A'CM_c.$$

In the Newton-Euler formulation, the contribution above must be multiplied by (see Appendix) B' and, therefore, takes the form:

$$A'CM_c.$$

Therefore, the Newton - Euler equations of this further constrained rigid body are

$$\dot{\Theta} = B(\Theta)\Omega \tag{12}$$

and

$$J1\dot{\Omega} = \breve{\Omega}J1(\Omega) + N - \breve{\mathcal{R}}A'G + A'CM_c \tag{13}$$

The system, represented by equation 13, is actually a second order system with one degree of freedom. Traditionally and classically, the above system has not been further reduced except in very special cases [15], [16]. We provide a general, novel and computer-aided reduction of these two equations to a second order system here. Simulations are carried out for both the irreduced and the reduced system in section 4. Let us define the scalar function of time w of equation 10 and its integral with respect to time: Ξ, respectively, as the velocity and position variables in the reduced system.

$$\Xi = \int_0^t w\,dt. \tag{14}$$

Let us define Θ as functions of this new position variable Ξ. This new relation, $\Theta(\Xi)$, will be numerically established. It follows, from equation 10 and equation 12 that

$$\partial\Theta/\partial\Xi = B(\Theta)A'\tau$$

These three nonlinear equations can be integrated under the following initial conditions: at $\Xi = 0$, the initial Θ is also zero. With the given initial conditions, one integrates the three nonlinear equations to derive the functions

$$\Theta = \Theta(\Xi). \tag{15}$$

This latter relation reduces the system to a second order system by projection, as shown in the Example. Analytical reductions may be possible by the use of integration tables [17] or when τ is parallel to a principal axis.

B.2 One Constraint

The same procedure as developed above for two constraints can be extended to the case of one constraint. Let τ be a 3×2 matrix that has two orthogonal unit vectors as columns. Let W be a two vector of proportionality constants; and let Ξ be the time integral of vector W. It can be shown that Θ as function of vector Ξ can be described by the nonlinear partial differential equation:

$$\partial\Theta/\partial\Xi = BA'\tau \tag{16}$$

with appropriate initial boundary conditions: when $\omega_1 = 0$, $\Theta = 0$; and, when $\omega_2 = 0$, $\Theta = 0$

In general, these partial differential equations (PDQ) are nonlinear, and numerical solutions seem to be the only method to arrive at solutions. This aspect of the current representation, namely, solution of the (PDQ), is not further pursued here.

III. Stability

The general problem of global stability of systems of free and constrained rigid body systems is not treated here. The stability of a single rigid body is considered in references [9], [8]. The stability of constrained and embedded rigid body systems is addressed in references [18], [8], [11]. For completeness, it is pointed out here, briefly, that with appropriate nonlinear position and velocity feedback, the global stability of these systems under the influence of gravity can be established. The total energy of the system is the Lyapunov function candidate.

Further, it can be shown that, for constrained systems, the restriction of the controls, and the Lyapunov function to the sub manifold of motion are, respectively, the appropriate control strategy and the Lyapunov function for global stability.

The example below demonstrates the formulation and stability for a single rigid body.

IV. Illustrative Examples

Two examples are presented in this section for demonstration of the methodology and stability and control.

A. Constrained System in Φ Space

In this example, a rigid body rotates about its center of gravity around an axis τ. The physical parameters of the system are given in section 7.4. The integration of the nonlinear differential equations of Θ as functions of Ξ results in the functions displayed in figure 1. There are two control mechanism: The first one is nonlinear feedback

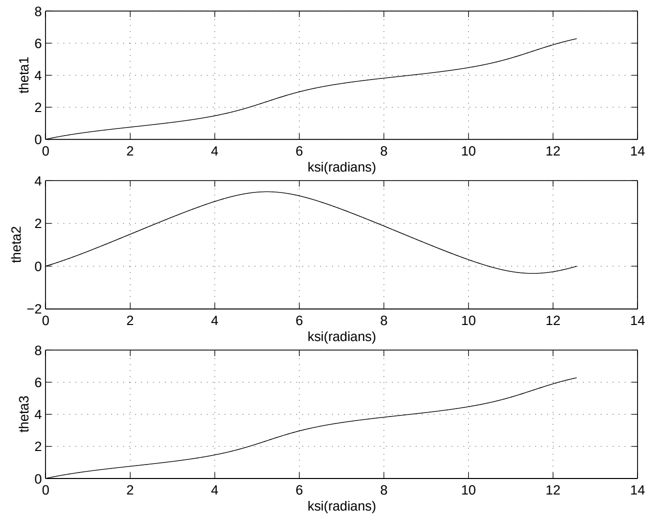

Fig. 1. The three angles Θ as functions of Ξ according to equation 15 where Ξ is the time integral of the angular velocity vector Ω according to equation 14.

that accomplishes global stability, limited by violation of the Lipschitz condition near

$$\Theta_2 = \pi/2.$$

The second control maintains the non-holonomic constraints without violating the stability of the system. Let the first control be as follows:

$$N = -B'(\Theta)KR_a(\Theta) - LR_a\Omega \qquad (17)$$

The second control applies impulsive control inputs when the non-holonomic constraints are about to be violated by a small quantity, as would be expected from a body rotating about a rigid axis. Let the magnitude of the two-vector impulsive torques of constraint be N_c. Let $\Omega-$ and $\Omega+$ be, respectively, the angular velocity vector immediately before and after the impulses are applied. It follows that

$$J[(\Omega+) - (\Omega-)] = A'CN_c.$$

As stated before, C is a 3×2 matrix whose columns are orthogonal to τ. In order for the $(\Omega-)$ to satisfy the constraint, N_c can be computed

$$N_c = -(C'AJA'C)^{-1}C'A\Omega. \qquad (18)$$

With the above two controls, the one rigid body is simulated from the initial conditions $\Theta = 0$ and $w = 16$:

$$[\Theta', \Omega'] = [0, 0, 0, 16\tau']$$

Two Lyapunov functions [8] are proposed for this example: the first is the stored energy of the unconstrained system. The second one is a modified energy function that has the same time -derivative as the first one.

These two Lyapunov functions, respectively, are:

$$v = 0.5\Omega' J\Omega + 0.5(\Theta)'KR_a(\Theta).$$

$$v2 = v - \int_0^t (\Omega' A' C N_c) dt. \qquad (19)$$

B. *The Reduced Ξ and w System*

The second example is a reduced case of example 1. the reduced one-degree of freedom system and the controls - being the projection of the controls of the first example - are given by the mappings:

$$\Theta = \Theta(\Xi)$$
$$\Omega = BA'\tau w. \qquad (20)$$

The Lyapunov function is the projection of the Lyapunov function of example 1 onto the Ξ and w space from equation 20.

C. *Digital Computer Simulations*

For the computational results presented here, each example was separately simulated, but the trajectories were superimposed in the next two figures. The trajectories of Θ, Ω versus time, and the trajectory of ω_2 and ω_3 versus ω_1 are shown in figure 2 both for the constrained system - solid line - and the reduced system - dotted lines.

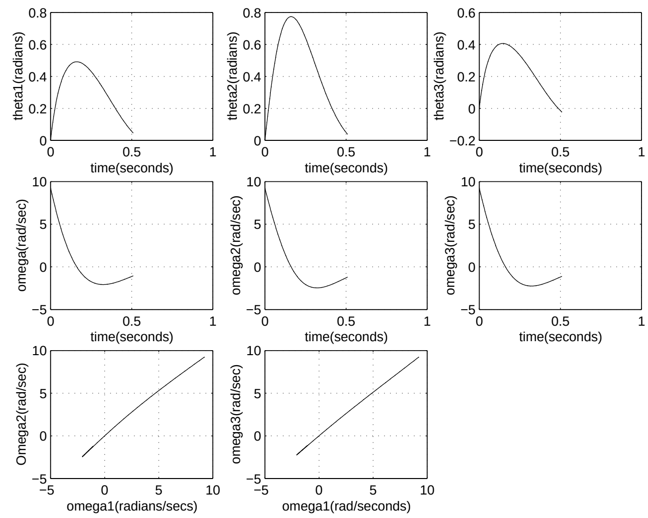

Fig. 2. The trajectories of Θ and Ω as functions of time; and Ω_2 and Ω_3 as functions of Ω_1 for the constrained and the reduced examples

The Lyapunov function v and its derivative with respect to time, for both examples, are shown in figure 3.

For elucidation and completeness, two more figures are presented. The two torques of constraint, in the constrained example, as functions of time, are shown in figure 4. Finally, the states Ξ and its derivative with respect to time w are shown in figure 5.

A comparison of the simulations for the two examples shows good correspondence.

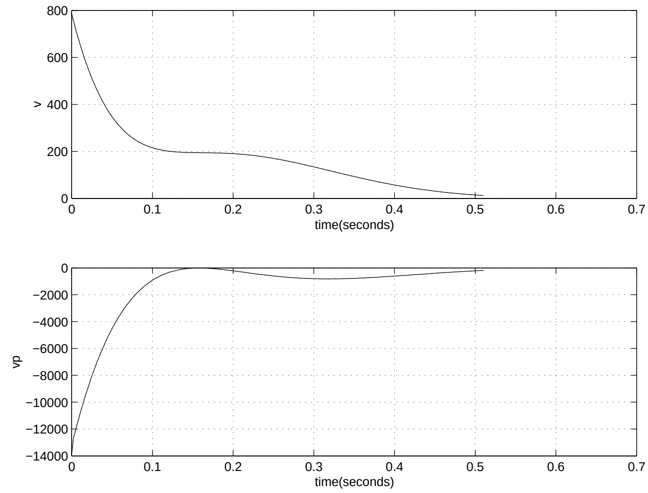

Fig. 3. The first Lyapunov function and its time rate as functions of time. Solid line is for the constrained case and the dotted line is for the reduced system. The superimposed trajectories are hardly distinguishable.

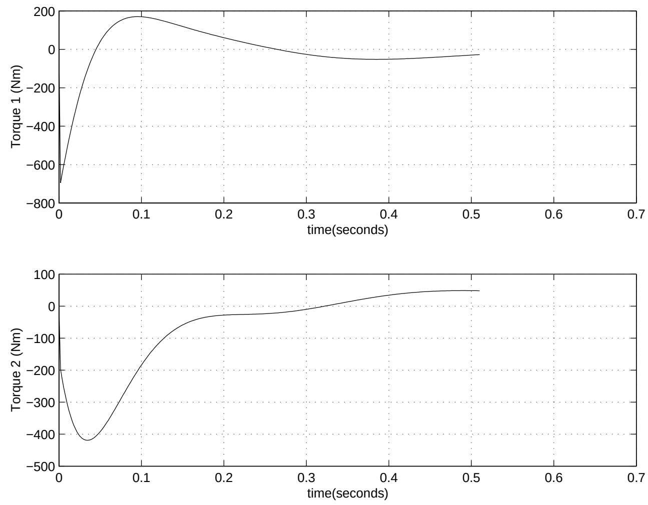

Fig. 4. The two nonholonomic torques of constraint as functions of time for example 1 - the constrined rigid body

V. Summary

A convenient state space formulation of rigid body dynamics is discussed that is modular and flexible. Its most salient feature is the uniformity and convenience of writing the equations of motion for stability, simulation and control design. It minimizes human effort, and relegates the computation load to the computer.

The methodology here can be extended to multi-linkage systems. An application to robots on the ground and in the air is carried out in reference [19]. For an application to human walking and turning, see reference [20]. An application to head and torso movement with muscles and neural control is carried out in reference [21]. Two examples in construction of Lyapunove function for planar and three-dimensional rigid body systems are in references [22], [23].

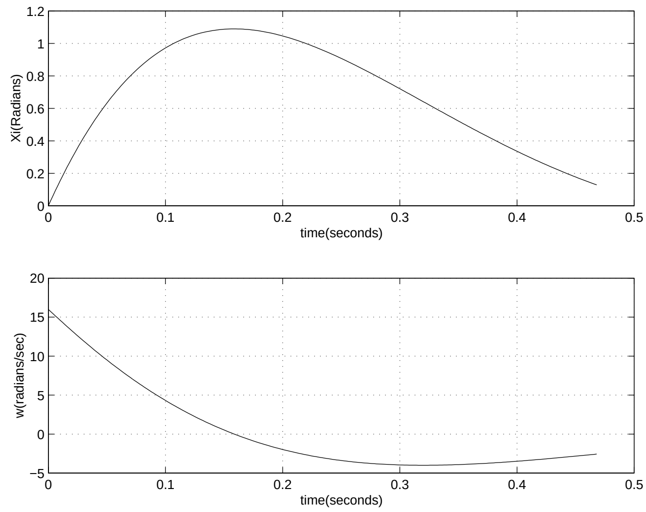

Fig. 5. The states Ξ,and w as functions of time.

VI. Dedication

This work is dedicated to the memory of Rajko Tomovic for his unrelenting vision, spirit, energy, pioneering research efforts and kindness.

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