A General Framework for Rigid Body Dynamics, Stability and Control

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Abstract

Augmented state spaces for the representation of systems that include rigid bodies, actuators, and controllers; and integrate mechanical, electrical, sensory, and computational subsystems, are proposed here. The formulation is based on the Newton - Euler point of view, and has many advantages in stability, control, simulation and computational considerations. The formulation is developed here for a one- and two-link three-dimensional rigid body system. Three simulations are presented to study stability of the system and to demonstrate feasibility and application of the formulation.

The formulation affords an embedding of the system in a larger state space. The rigid body system can be stabilized, in the sense of Lyapunov, in this larger space with very general and minimally restricted feedback structures. The formulation is modular to implementation and is computationally efficient. The method offers alternative states that are easier to control and measure than Euler angles. Thus, the formulation offers advantages from a sensory and instrumentation point of view. The formulation is versatile, and yields conveniently to applications in studies of human neuro-musculo-skeletal systems, robotic systems, marionettes and humanoids for animation and simulation of crash and other injury prone maneuvers and sports. It offers a methodical and systematic procedure for formulation of large systems of interconnected rigid bodies.

Keywords: rigid body dynamics, augmented state space, Lyapunov stability, embedding, control, simulation, coupling, nonlinear control, large systems

1 Introduction

This paper is concerned with the concepts of rigid body dynamics, stability and control. The concepts are addressed within the framework of a one- and two-link three-dimensional rigid body system [1–4]. While each of the above concepts merit separate treaties [5–9], the presentation here emphasizes an integrated point of view of all three [10]. This integrated point of view is useful in formulating the dynamics, in computer simulation and animation [4, 11, 12] of natural and artificial systems, and in modeling humanoid, robotic and marionette systems [13–15] that are comprised of systems of connected rigid bodies. Other applications in assistive devices [16] and in crash simulation systems can be cited. Another important application of this approach is in the realm of miniaturized electro-mechanical system where mechanical components, electrical drives, instrumentation, computational units and interfaces are densely packaged in small spaces. As robots, manipulators, and probes become smaller [17, 18] and more self-contained, such integration of all the system components becomes more desirable.

As part of the formulation, every single rigid body of the system, with its six degrees of freedom, is embedded in a nine-degree- of-freedom space. The stability of the system is established in this nine-dimensional space - an eighteen dimensional state space. Additional states may be needed for the construction of Lyapunov functions, and hence the state space of every rigid body is approximately of dimension 20 rather than 12. The state augmentation technique [19], on its own, merits further study; however, this subject is not further pursued here.

Two attributes of the embedding are a new state space and partial control of the trajectories as in the sliding mode method [20]. In a broader sense, the addition of the three degrees of freedom affords design flexibility and implementation ease. In this newly constructed larger state space, the tasks of measurements and instrumentation are easier. A methodology is outlined here in which other needed variables, such as the Euler angles [21–23], can be indirectly arrived at, i.e., computed rather than directly measured.

The method encompasses several recent developments: passive and active interaction with the environment, and integration of control, instrumentation, and computation with the system dynamics. Additional attributes are: modularity of implementation and simulation, ease of representation of rigid body systems of large dimension, and stability studies. Of course, conventional linearization, pole assignment, and decoupling strategies [24] apply equally here.

The structure of the system, and the extension of this method to many rigid bodies, is very convenient for Lyapunov stability studies [8, 9, 25–27]. Of particular importance is that the energy of the system, the sum of elastic, kinetic, and potential energies and variations of the energy formula, are the Lyapunov function candidates. Analytical proof of asymptotic stability can be provided by the use of the perturbation approach [28]. The analysis is facilitated by the quadratic forms in the energies of the system and the particular feedback structures utilized here. These forms afford the application of several conventional matrix norms in order to prove the asymptotic stability of the system. Direct and indirect control methods [29] and other methods, such as in reference [30] for constrained robotic systems, also can be utilized in stability studies. These analytical proofs, however, are not advanced here. The present approach is not to emphasize the analytical and mathematical aspects. Instead, the emphasis in this paper is on the formulation, simulation, and computations necessitated by applications, need for modularity, expandability and large dimensionality of the systems involved.

In section 2, the dynamics and the state space representations of a single rigid body are presented. In section 3, the issues of stability and control are addressed, and the Lyapunov method is used in order to establish stability of the one- and two-link rigid body systems. Three simulations of the motion of a one- and a two-link rigid body system are presented in section 4. Discussion of applications and conclusions are drawn in section 5. Appendix and references follow.

2 The Single Rigid Body

In this section, three issues are discussed. First, the dynamics of a single rigid body is formulated in a simple state space. Augmented state space of one rigid body, and certain applications are considered next. Finally, the simple state space is generalized for ease of measurement and instrumentation.

2.1 State Space Formulation

Consider a single rigid body under the influence of gravity, a translational force Λ, expressed in an inertial coordinate system (ics), and a rotational input couple N which is expressed in the body coordinate system (bcs) [31,32]. For simplicity and ease of notation, all symbols and definitions are borrowed from [32]. Let the mass of the body be m and its moment of inertia matrix J, expressed in the principal axes coordinate system of the body, be the diagonal 3 by 3 matrix whose elements are given in the Table 1 of the Appendix. The origin of this body coordinate system is at the center of gravity of the body. The inertial coordinate system, relative to a biped, is defined as a right-handed coordinate system as follows: x_1 axis to the front, x_2 axis towards the extended left arm, and the x_3 axis vertically upward [33]. The definitions of the physical parameters, their symbols, and the numerical values of the parameters for the simulations are given in the Appendix.

Let three-component vectors X and V define the translational position and velocity of the center of gravity of the rigid body in the inertial coordinate system (ics). Let R be the coordinates of the contact point of force Λ in the body coordinate system (bcs). Let Θ and W be respectively the Euler angles and the angular velocity vector of the body in bcs [32]. Let $\check{R}$, A, and B be the 3 by 3 matrices as defined in the Appendix. Let $M = diagonal(m, m, m)$. The conventional state space equations of the system are Θ and $\dot{\Theta}$ [24,34]. A more convenient state space formulation [31] is given by

$$\begin{aligned} \dot{X} &= V \\ M\dot{V} &= \Lambda + G \\ \dot{\Theta} &= B(\Theta)W \end{aligned}$$

$$J\dot{W} = f(W) + N + \breve{R}A'\Lambda, \tag{1}$$

where f is a nonlinear algebraic function [31,32] and is given by

$$f = -\breve{W}JW \tag{2}$$

It is worth noting that when $R = 0$, the two subspaces of rotation and translation are decoupled.

2.2 Augmented State Space and Applications

In order to stabilize the above rigid body, a passive feedback structure that includes two elastic and one viscous element per degree of freedom is proposed. This amounts to embedding the rigid body state space in an augmented state space as follows. With the help of six additional degrees of freedom, D for translation and Φ for rotation, the feedback structure is defined. Let KT_a, KT_b, LT_a, LT_b, KR_a, KR_b, LR_a and LR_b be eight 3 by 3 positive definite matrices. The passive feedback dynamics are given by

$$\begin{aligned} \Lambda &= -KT_a(X - D) - LT_aV \\ KT_a(X - D) - KT_bD &= LT_b\dot{D} \\ N &= -B'(\Theta)KR_a(\Theta - \Phi) - LR_aW \\ KR_a(\Theta - \Phi) - KR_b\Phi &= LR_b\dot{\Phi}. \end{aligned} \tag{3}$$

The embedded (augmented) state space for a rigid body and its controller are defined by equations 1 and 3. Suppose the above control is to be implemented strictly by passive structures, i.e., translational and rotational linear springs and linear viscous friction elements, commonly referred to as shock absorbers or dashpots [35]. By reducing equation 1, for a point mass with one translational degree of freedom, one obtains the system shown in figure 1a. In this figure, the bottom line defines the ground or, alternatively, the inertial coordinate system. The structure consists of four passive elements: two springs and two dashpots connected in the particular configuration shown – a parallel visco elastic element connected serially to another parallel visco-elastic element. The values of the elements can be selected to make the system of figure 1a underdamped, overdamped or critically damped. Viewed in this light, the augmented rigid body system affords passive interaction with the environment [12,19,36]. In the present formulation, the augmented states correspond to position states. In [19], the augmented states correspond to the velocity states by which a mechanical manipulator is controlled. An alternative view of passive position state control can be attained from the work of Lee and Shin [12] in graphical animation. A system of elastic nets surrounds a graphical "mass-less" biped to allow its smooth control and movement on the screen. The passive structure implementation of the augmented state space can offer alternative implementation of a smoothing and stabilizing structure for graphical applications, where the biped possesses massive body parts and is under gravity and other forces from muscles and external sources, and interacts with the environment through physical and measurable forces.

2.3 Applications

The augmented state space is useful in a variety of applications: inclusion of actuator dynamics, convenience of measurement and instrumentation, stability and control, etc.

2.3.1 Musculoskeletal Formulation

In mathematical modeling of living systems and biorobots, there are a large number of muscles or actuators to be included. The implementation of an actuator in the musculoskeletal system can be envisioned in figure 1b. The load describes one bone - the origin of the muscle attachment. The connection describes the insertion point of the muscle on another bone [36,37]. The element K is the tendon. The active element is a muscle model with neural input and force output. The element L is the viscous element of the surrounding tissues. However, it must be remembered that natural muscles and tendons are unidirectional force generators, i.e., they can pull but cannot push. Nevertheless, the augmented rigid body model here allows one to model physiological systems that include passive tissue such as ligaments, cartilage, and active muscle models that include tendons.

2.3.2 State Space Transformations

The augmented state space allows flexibility of choosing other state spaces that are convenient for instrumentation, indirect measurement, computational efficiency, etc. Two examples are introduced here, one for the rotational subspace and one for the translational subspace.

Suppose a couple τ is defined as below, and that this couple is easy to measure.

$$\tau = KR_a(\Theta - \Phi) \tag{4}$$

One can select τ, Φ and W as a new set of position states. In the latter state space, the input torque: N and the Euler angles Θ turn out to be functions of the state, and hence can be treated as outputs of the system, i.e., sums of the states: τ, Φ and W. The new state space equations are:

$$\begin{aligned} J\dot{W} &= f - B'(\Theta)\tau - LR_aW \\ \dot{\tau} &= KR_aB(\Theta)W - KR_a(LR_b)^-1(\tau - KR_b\Phi) \\ \dot{\Phi} &= (LR_b)^-1(\tau - KR_b\Phi) \end{aligned} \tag{5}$$

The output equations are:

$$
\begin{aligned}
\Theta &= (KR_a)^-1(\tau) + \Phi \\
N &= -B'\tau - LR_a W.
\end{aligned}
\tag{6}
$$

Since τ is a torque quantity, there are situations where it is easier to measure τ than the Euler angles Θ. This allows flexibility in arriving at Θ by computation. For example, in DC motors, the produced torque is proportional to the armature current [23, 38] which, from an instrumentation point of view, is easy to sense.

The translational subsystem is Example 2. An analogous reasoning can be extended to this part of the system of equation 1, and is briefly summarized below. Let β be a vector force as defined below:

$$
\beta = KT_a(X - D) \tag{7}
$$

The state space equations for the translational subsystem are

$$
\begin{aligned}
M\dot{X} &= G - \beta - LT_a V \\
\dot{\beta} &= KT_a W - KT_a(LT_b)^-1(\beta - KT_b D) \\
\dot{D} &= (LT_b)^-1(\beta - KT_b D)
\end{aligned}
\tag{8}
$$

The output equations are:

$$
\begin{aligned}
X &= (KT_a)^-1\beta + D \\
\Lambda &= -\beta - LT_a V.
\end{aligned}
\tag{9}
$$

2.4 Generalized State Space Formulation

It is, sometimes, desirable to replace the coordinates of the center of gravity with coordinates of certain points where the rigid body is in contact with external forces, objects, or other points whose coordinates are more easily accessible for instrumentation and measurement [22]. Let the coordinates of such a point be X_c in the ics and vector Q in the bcs. It follows that

$$
\begin{aligned}
X_c &= X + AQ \\
\dot{X}_c &= \dot{X} - A(\breve{Q})W \\
\ddot{X}_c &= \ddot{X} - A(\breve{Q})\dot{W} + A(\breve{W})^2 Q.
\end{aligned}
\tag{10}
$$

From the latter point of view, the state space equations of the system about an arbitrary point on the body, rather than the center of gravity, can be derived. These equations are given below.

$$
\begin{gathered}
\dot{XC} = VC. \\
\dot{\Theta} = B(\Theta)W. \\
\begin{bmatrix} \dot{VC} \\ \dot{W} \end{bmatrix} = \begin{bmatrix} M^{-1} & -A\check{R}J^{-1} \\ 0 & J^{-1} \end{bmatrix} \begin{bmatrix} \Lambda + G + MA(\check{W})^2 R \\ f - N + \check{R}A'\Lambda \end{bmatrix}. \\
\dot{D} = LT_b^{-1}(KT_a(XC - D) - KT_b D) \\
\dot{\Phi} = LR_b^{-1}(KR_a(\Theta - \Phi) - KR_b \Phi).
\end{gathered}
\tag{11}
$$

For the feedback variables, one has two choices: either the coordinates of the center of gravity or those of the point of contact can be utilized. For the latter case, the stabilizing feedbacks are:

$$
\begin{gathered}
\Lambda = -KT_a(XC - D) - LT_a VC \\
N = -B'(\Theta)KR_a(\Theta - \Phi) - LR_a W.
\end{gathered}
\tag{12}
$$

As noted before, in the first representation, i.e., equations 1 and 3, with $R = 0$, the two subspaces of rotation and translation are decoupled. When $R \neq 0$, there is a one directional coupling from the translational subspace into the rotational subspace. When the coordinate system is with respect to a moving point, there is bidirectional coupling between the two subspaces.

Suppose a point of the body, with coordinates Q in the bcs, is fixed in the ics coordinate system. The equations of motion, for this case, are formulated in references [5,32], and, as a consequence of the fixed point, the dimension of the state space above can be reduced [31,32] from 18 to 9.

3 Stability

3.1 Single Rigid Body

It is shown in this section that the system of equations 1 and 3 is asymptotically stable about the origin except near $\theta_2 = 0.5\pi$ where $B(\Theta)$ becomes singular. Lyapunov methods can be utilized for proving asymptotic stability by construction of suitable Lyapunov functions.

The proof of asymptotic stability is sketched here briefly. The perturbation approach ([28], Chapter 5) can be used effectively here due to the quadratic form of the energies of the system and the particular positive definite feedback structures proposed here. These forms afford application of several convenient matrix norms in order to prove the asymptotic stability of the system. In order to keep the presentation simple, at first the gravity effect is ignored, and the development is limited only to the system of equations 1 and 3.

Let v_T and v_R be the total energy stored in each of the rotational and translational subsystems under the conditions that they were not coupled by the term $\check{R}A'W$. These energies are respectively

$$
\begin{aligned}
v_T &= 0.5V'MV + 0.5(X-D)'KT_a(X-D) + 0.5D'KT_bD \\
v_R &= 0.5W'JW + 0.5(\Theta-\Phi)'KR_a(\Theta-\Phi) + 0.5\Phi'KR_b\Phi.
\end{aligned}
\tag{13}
$$

Let $\dot{\epsilon}$ be the upper bound of the power (instantaneous energy) transferred from the translational subsystem to the rotational subsystem, i.e., let

$$
\begin{aligned}
\dot{\epsilon} &= W'\check{R}A'\Lambda \\
\epsilon &= \int_0^t W'\check{R}A'\Lambda dt.
\end{aligned}
\tag{14}
$$

Now the overall Lyapunov function for the system of equations 1 and 3 is defined by

$$v = v_T + v_R - \epsilon. \tag{15}$$

It can be shown, due to the quadratic nature of the above function v, that it satisfies the necessary conditions for v to be a Lyapunov function. Further, by differentiation of equation 15 with respect to time and making use of equations 1 and 3, one can show that $\dot{v}$ is negative semi-definite, i.e.,

$$\dot{v} \leq 0.$$

It therefore follows that the system is asymptotically stable. The formal Lyapunov proofs are left out. By construction, the eight positive definite matrices of equation 3 can always be selected such that all the necessary conditions for v and $\dot{v}$ are satisfied.

It is to be noted that when $R = 0$, the Lyapunov function is merely the sum of the energies, elastic plus kinetic of the two decoupled rotation and translation subspaces.

3.2 Constrained Rigid Body

When the rigid body is holonomically constrained by being, for example, in contact with the origin of the ics, two cases arise. Either the constraint is permanently in effect or the constraint can be violated under control.

In the first case, the equations can be reduced in dimension, i.e., the motion can be projected onto the manifold of constraint [31, 32]. The most important attribute of the present formulation is that the same projection can be applied to the controls and the Lyapunov functions. It can be shown that the these projections are, respectively, the stabilizing control and the corresponding Lyapunov function for the reduced system. This attribute allows for a systematic and step by step method of designing feedback and Lyapunov functions for holonomicaly constrained rigid body systems such as present day multi-link robots, etc. Further discussions of this issue, and a numerical example are presented in reference [15]. This development extends the previous applications of Lyapunov methods in linear systems [34] to systems of unconstrained and constrained rigid bodies.

In the second case, the same method of projection of the Lyapunov function can be utilized provided that the motion of the system, a priori, is guaranteed to lie on the constraint manifold. Maintaining the system's trajectory on the constraint manifold requires additional conditions on the eight control matrices. The details are not given here. The more interesting application of the latter point of view is to extend it to large systems of rigid bodies where suitable controls and Lyapunov functions are design objectives.

When gravity vector G is present in equation 1, two additional steps are necessary: 1). The position state variables X, D, and, generally speaking, Θ and Φ, have to be biased initially to account for G; 2). The gain matrices in equation 3 have to be large enough that any loss of potential energy, due to disturbance and perturbation of the system, is more than compensated by the energy stored in the elastic elements [36].

When the system is capable of being propelled into the air [1, 39], the potential energy, due to the rise or fall of the center of gravity, has to be added to the Lyapunov function. The implementation of gravity in a hopping motion is demonstrated later in simulation 2.

It is instructive to compare the above very general method of constructing stabilizing feedbacks and Lyapunov functions to previous efforts [25–27, 32] and to previous applications of the Lyapunov method to the inverted pendulum systems [2, 9, 20, 28]. The present formulation is suitable for large systems, involves vectors and matrices rather than individual Euler angles, and scaler gains, and is very convenient for computer implementation.

3.3 Multiple Rigid Bodies

In this section, two stable rigid bodies are coupled, and the system's stability is considered. The extension to several coupled bodies is obvious [31]. In order to simplify the notation and presentation, a more compact and symbolic form of equations 1 and 3 is first presented.

Let Z designate the state of one rigid body:

$$Z' = [X', V', D', \Theta', W', \Phi'].$$

With Z thus defined, equations 1 and 3 can be symbolically written as

$$H\dot{Z} = F(Z)$$

Here, H is an 18 by 18 matrix, that for convenience and ease of mathematical notation, has the following diagonal elements: I_3, M, I_3, I_3, J, I. Here, I_3 defines a 3 by 3 identity matrix. With this definition of H, the inverses of LT_b and LR_b are integrated in $F(Z)$. Let μ and the Lyapunov function v be defined as before.

Let Δ be an external force acting on the body at a point with coordinates S in bcs. The dynamics of the system are described by

$$H\dot{Z} = F(Z) + G(Z)\Delta.$$

Here, $G(Z)$ is an 18 by 3 matrix whose entries are the six 3 by 3 matrices, i.e.,

$$G(Z) = [0, I_3, 0, 0, A\breve{S}', 0]'.$$

With the above definitions in hand, the two rigid body system is defined below by

$$\begin{aligned} H_1\dot{Z}_1 &= F_1(Z_1) + G_1(Z_1)\Delta_{12} \\ H_2\dot{Z}_2 &= F_2(Z_2) - G_2(Z_2)\Delta_{12}. \end{aligned} \tag{16}$$

In this formulation, the force Δ_{12} can be, in general, described by

$$\Delta_{12} = \Delta_{12}(Z_1, Z_2, \mu) \tag{17}$$

Here, μ describes augmented states of the active or passive coupling between the rigid bodies. Two different passive cases are considered in the simulation section, and one active case is discussed in the discussion section.

Let

$$v_1(Z_1, \epsilon_1)$$

and

$$v_2(Z_2, \epsilon_2)$$

be, respectively, the two Lyapunov functions, as previously defined, for the two rigid bodies. Let σ_1 and σ_2 be additional states defined by

$$\begin{aligned} \dot{\sigma_1} &= \dot{Z_1}'G_1(Z_1)\Delta_{12} \\ \dot{\sigma_2} &= \dot{Z_2}'G_2(Z_2)\Delta_{12}. \end{aligned} \tag{18}$$

Let v_{12} be a candidate for the Lyapunov function of the two rigid body system.

$$v_{12}(Z_1, Z_2, \epsilon_1, \epsilon_2, \sigma_1, \sigma_2) = v_1 + v_2 - \sigma_1 + \sigma_2) \tag{19}$$

It can be shown that

$$\dot{v_{12}} = \dot{v_1} + \dot{v_2}. \tag{20}$$

Therefore, $\dot{v}_{12}$ is negative semi-definite. If v_{12} satisfies the conditions for Lyapunov function, it is a proper Lyapunov function; and the system of two rigid bodies is asymptotically stable. This is demonstrated below in simulation 3.

4 Simulations

The results of three simulations are reported in this section: transient of a single rigid body, hopping motion of a single rigid body, and transient of a two body system under a passive coupling structure.

4.1 Single Rigid Body Transient

The system parameters for this simulation are given in the Appendix. The system is described by the following equations:

$$
\begin{aligned}
\dot{XC} &= VC. \\
\dot{\Theta} &= B(\Theta)W. \\
\begin{bmatrix} \dot{VC} \\ \dot{W} \end{bmatrix} &= \begin{bmatrix} M^{-1} & -A\breve{R}J^{-1} \\ 0 & J^{-1} \end{bmatrix} \begin{bmatrix} \Lambda + G + MA(\breve{W})^2 R \\ f - N + \breve{R}A'\Lambda \end{bmatrix}. \\
\dot{D} &= LT_b^{-1}(KT_a(XC - AR' - D) - KT_b D) \\
\dot{\Phi} &= LR_b^{-1}(KR_a(\Theta - \alpha\Phi) - KR_b\Phi)
\end{aligned}
\tag{21}
$$

With the feedback, one has two choices: either the coordinates of the center of gravity or those of the point of contact are utilized. For the latter case:

$$
\begin{aligned}
\Lambda &= -KT_a(XC - AR' - D) - LT_a(VC + ARRW) \\
N &= -B'(\Theta)KR_a(\Theta - \alpha\Phi) - LR_a W.
\end{aligned}
\tag{22}
$$

Several remarks are in order here.

1. The translation coordinates XC and VC here are those of the force contact point rather than those of the center of gravity.
2. The feedbacks utilized in Λ involve the coordinates of the center of gravity rather than those of the point of contact.
3. In order to speed up the transient decay rate, the variables Φ are amplified by a factor of $\alpha = 0.2$.
4. In order to account for gravity, the third translational state and the corresponding augmented position state are set, respectively, to

$$XC0 = [0, 0, -0.67]'$$

and

$$D0 = [0, 0, -0.125]'$$

meters. The Lyapunov function accounts for the initial compression of the elastic elements due to gravity. This means:

$$v_T = 0.5(VC + A\breve{R}W)'M(VC + A\breve{R}W) \tag{23}$$

$$+0.5(XC - D - (XC0 - D0))'KT_a(XC - D - (XC0 - D0)) \\ +0.5(D - D0)'KT_b(D - D0). \tag{24}$$

Lyapunov function v, computed from equation 15, its rate $\dot{v}$, the function ϵ and its rate $\dot{\epsilon}$ as functions of time are, respectively, shown in figure 2. The nine translational state trajectories are plotted versus time in figure 3. The nine rotational state trajectories are plotted versus time in figure 4. The system, initially disturbed by

$$W0 = [8, 16, -8]',$$

returns to its equilibrium in about two seconds. The simulation also shows that the rotational degrees of freedom track the augmented states relatively better than the translational states.

4.2 Hopping Motion of One Rigid Body

In this simulation, the parameters of the single rigid body are adjusted such that, through storage of sufficient elastic energy, the rigid body is propelled into air, falls back down, and, through conversion of the kinetic energy to elastic energy, the system is compressed. This energy, due to compression, again lifts the system to lesser height and so forth [6, 39, 40]. Depending on the amount of the initial energy stored, a sequence of hops or bounces occur, as shown in figure 5. During the time that the system is in the air, the translational Lyapunov function, i.e., the sum of the kinetic and potential energy of the system is constant. The overall Lyapunov function decreases in this phase due to the continuous dissipation of rotational energy. The height of the center of gravity, as a function of time, is plotted in figure 5.

4.3 Two Rigid Body System Transient

In this simulation, two identical rigid bodies with the same parameters as in simulation 1 are coupled by a two-component linear viscous and elastic coupling. For simplicity, the gravity is assumed to be zero. The vectors R_1 and R_2 are equal to zero. Therefore ϵ_1 and ϵ_2 are both zero. The parameters K_c and L_c of this coupling are given in the Appendix. Initially, the two rigid bodies are given high angular velocity disturbances assumed to arrive from high impacts:

$$W_1 = [-8, -16, 8]'$$

and

$$W_2 = [8, 16, -8]'.$$

The Lyapunov function is taken to be the sum of the elastic and kinetic energies of the two bodies. The energy of the elastic coupling between the two rigid bodies is not included in the Lyapunov function. The Lyapunov function and its derivative with respect to time are plotted in figure 6. The 36 states of the system are plotted in four figures. The nine translational states of body 1 are plotted as functions of time in figure 7. The graphs in the first row show the three position states. The graphs of the second row show the three velocity states. The graphs of the last row show the three augmented states of D. The nine rotational states of body 1 are plotted as functions of time in figure 8. The first row graphs show the three position states. The second row graphs show the three angular velocities of the body W_1, i.e., the velocity states. The last row shows the three augmented states of Φ. The nine translational states of body 2 are plotted as functons of time in figure 9. The first row shows the three position states. The second row shows the three velocity states. The last row shows the three augmented states of D. Finally, the rotational states of body 2 are shown in figure 10

5 Discussion and Conclusion

In this paper, a novel method of representing coupled rigid bodies was presented. This method encompasses several recent developments. Passive and active interaction with the environment can be conveniently represented. The integration of control, instrumentation, and computation with the system dynamics is easily implemented. The modularity of implementation, and simulation, ease of representation of rigid body systems of large dimension and stability studies are other positive attributes. This method of representation also allows implementation of concepts for impedance control and measurement as in [41, 42]. An important attribute of the method is that it enables one to substitute other measurements and compute rather than measure Euler angles.

More generally speaking, the rigid body system could be embedded in larger dimensional state spaces where positions, velocities, accelerations, etc., could be represented, and which could have a role in control implementation and trajectory control. Alternative embedding strategies can be envisioned that constrain the trajectories of the rigid body system to lie on predefined and desired manifolds. This latter method complements the sliding mode methods of Utkin [20] and those of McClamroch and Wang [30].

To summarize, for systems of coupled rigid bodies, new state spaces have been discussed and formulated that are capable of integrating instrumentation, computation, and expansion. The method also allows integration of electrical, mechanical and control components. Lyapunov stability of these systems, with the help of energy functions, is systematic, easy to carry out, and, because of the generality of quadratic forms and their norms, amenable to rigorous analysis. The proposed feedback configurations are well structured and allow further refinements, optimization, or extension to constrained, constrainable, systems of variable structure, and inverse systems. Most importantly, stabilizing feedbacks and Lyapunov functions can systematically be constructed for holonomic systems that are projections of free rigid body systems. The projections of the feedbacks, and the Lyapunov function of the unconstrained free rigid body system are the corresponding controls and the Lyapunov function for the constrained system.

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6 Appendix

The matrices $A(\Theta), B(\Theta)$ and RR that appear in the formulation of single rigid bodies are given below.

Let $A_1(\theta_1), A_2(\theta_2)$ and $A_3(\theta_3)$ be defined by

$$A_1(\theta_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}.$$

$$A_2(\theta_2) = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -sin\theta_2 & 0 & \cos\theta_2 \\ & & \end{bmatrix}.$$

$$A_3(\theta_3) = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now $A(\Theta)$ can be defined:

$$A(\Theta) = A_1(\theta_1)A_2(\theta_2)A_3(\theta_3) \tag{25}$$

The matrix $B(\Theta)$ is given by:

$$B(\Theta) = \begin{bmatrix} \frac{cos\theta_3}{cos\theta_2} & \frac{-sin\theta_3}{cos\theta_2} & 0 \\ sin\theta_3 & cos\theta_3 & 0 \\ \frac{-sin\theta_2 cos\theta_3}{cos\theta_2} & \frac{sin\theta_2 sin\theta_3}{cos\theta_2} & 1 \end{bmatrix}. \tag{26}$$

Let vector R have components r_1, r_2, and r_3. The skew symmetric matrix $\breve{R}$ is defined as:

$$\breve{R} = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}. \tag{27}$$

As stated in the text, the moment of inertia matrix J, expressed in the principle axes coordinate system of the body, is diagonal:

$$J = diag[j_1, j_2, j_3].$$

The numerical values are given in the table below. Similarly, for vectors W, Q and S, skew symmetric matrices $\breve{W}$, $\breve{Q}$, and $\breve{S}$ are defined. The definitions, symbols, and numerical values for a one rigid body system [3, 32] are given in Table 1.

The one rigid body parameters for simulation 1 are $R = [0, 0, -0.42]'$
$KT_a = Diag(400, 400, 320)$
$KT_b = Diag(400, 400, 320)$
$LT_a = Diag(20, 20, 14)$
$LT_b = Diag(200, 200, 140)$
$KR_a = KR_b = Diag(4100, 4100, 3280)$
$KR_b = Diag(4100, 4100, 3280)$
$LR_a = Diag(60, 60, 42)$
$LR_b = Diag(600, 600, 420)$
$\alpha = Diag(0.2, 0.2, 0.2)$

The parameters in simulation 2 are:
$m = 20.5Kg$
$KT_a = Diag(410000, 41000, 32800)$
$KT_b = Diag(41000, 41000, 32800)$
$LT_a = Diag(200, 200, 140)$
$LT_b = Diag(200, 200, 140)$
The remaining parameters are the same as for simulation 1.

The additional parameters for simulation 3 are
$K_c = Diag(2000, 2000, 2000)$
$L_c = Diag(200, 200, 200)$

Table 1: Definition, symbols and numerical values for one rigid body.

	symbol	value	unit
mass,	m	41.00	Kg
p.m. of i.,	j1	10.0	Kg m^2
p.m. of i.,	j2	8.0	Kg m^2
p.m. of i.,	j3	0.4	Kg m^2
c. of g.,	l	0.42	m
gravity	g	10	m/s^2

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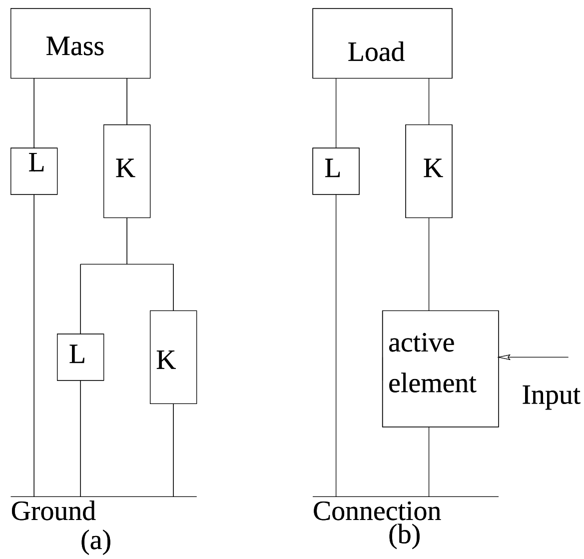

Figure 1: Passive and active components for a single degree of translation motion of a point mass. The variable K and L, respectively, refer to a linear elastic and viscous elements.

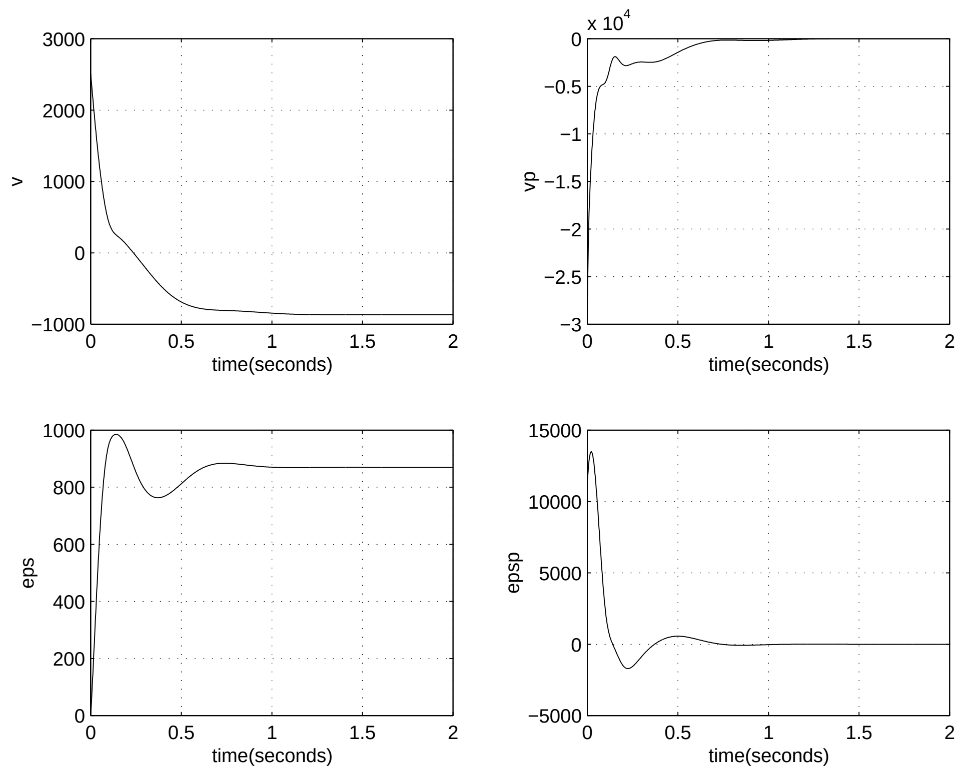

Figure 2: The Lyapunov function v, its rate $\dot{v}$, the function ϵ and its rate $\dot{\epsilon}$ as functions of time in simulation 1.

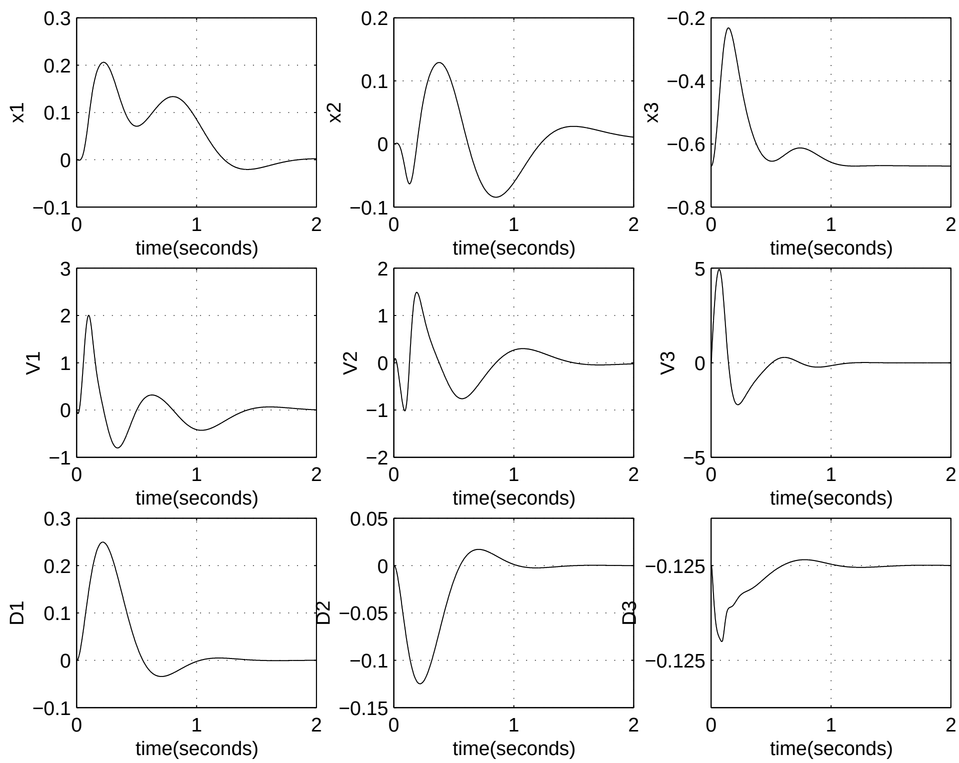

Figure 3: The nine translational state trajectories plotted versus time in simulation 1.

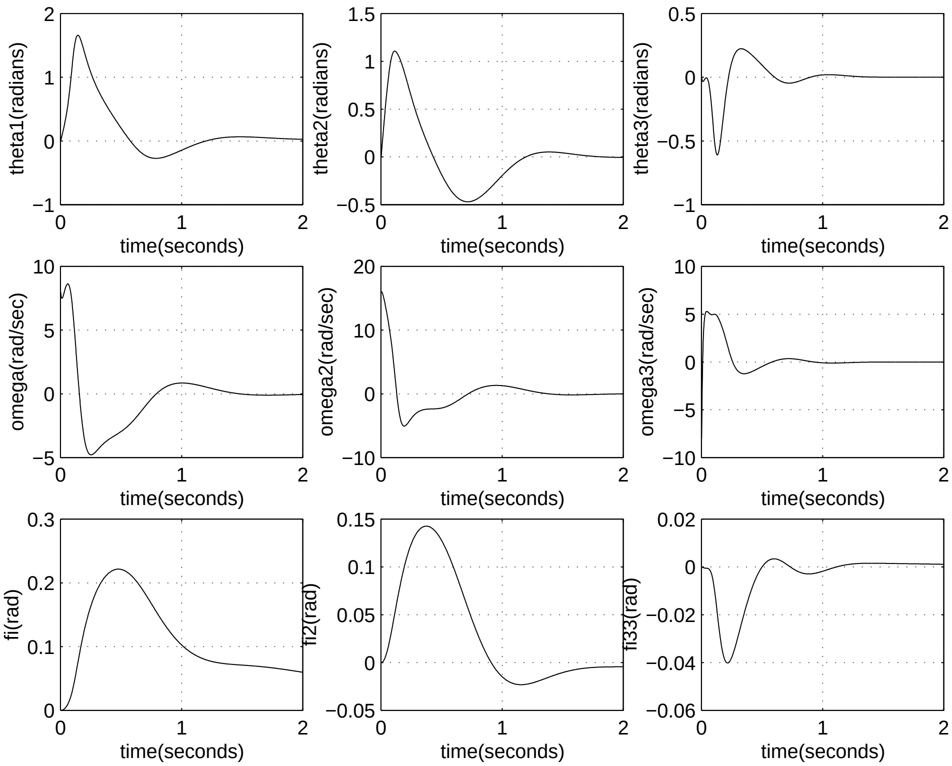

Figure 4: The nine rotational state trajectories plotted versus time in simulation 1.

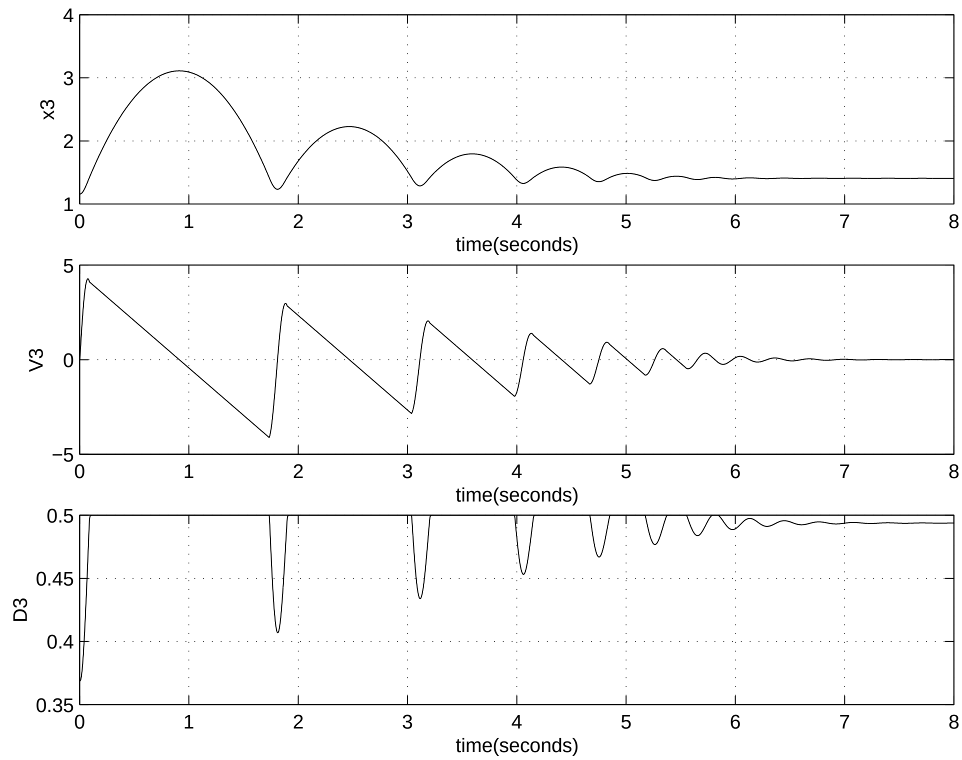

Figure 5: The vertical hopping of the one rigid body from simulation 2. The vertical height of the center of gravity, its velocity, and the corresponding augmented state are plotted as functions of time in simulation 2.

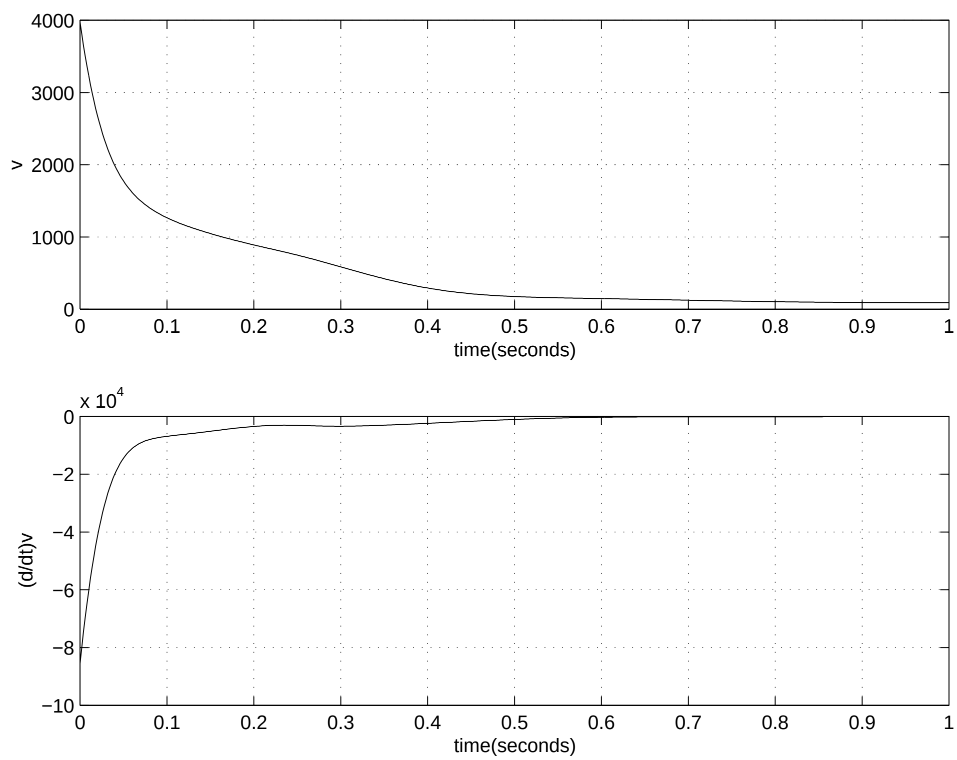

Figure 6: The Lyapunov function and its derivative as functions of time for a two-link rigid body system. The Lyapunov function is the sum of the translational and rotational energies of the two bodies in simulation 3.

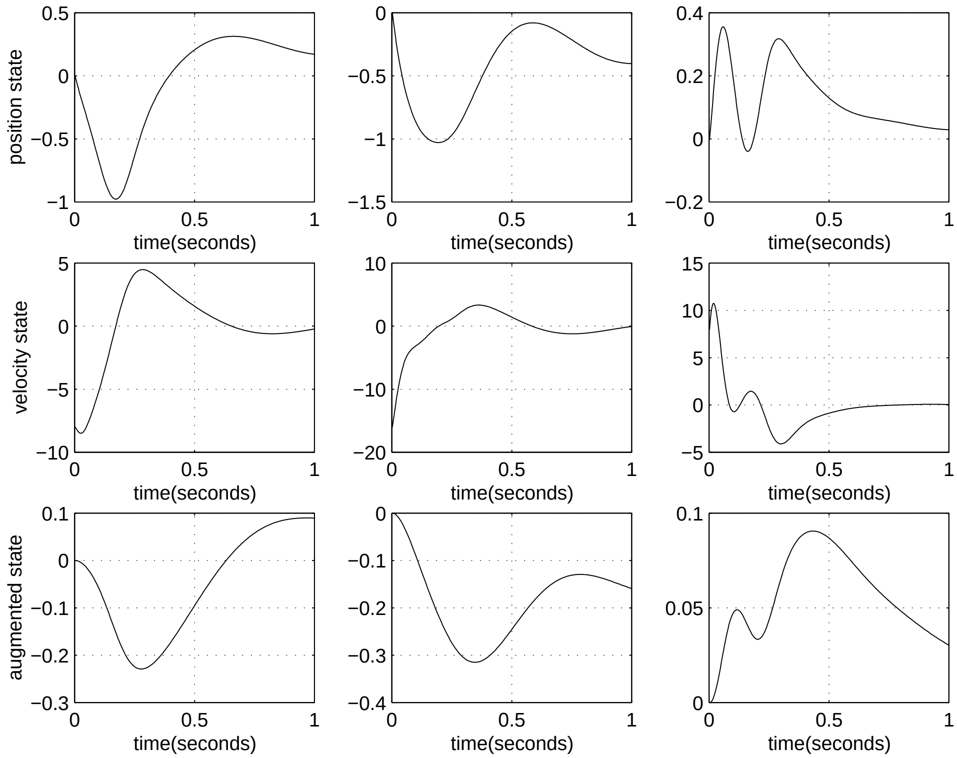

Figure 7: The nine state trajectories as functions of time for the rotational motion of body 1 in simulation 3.

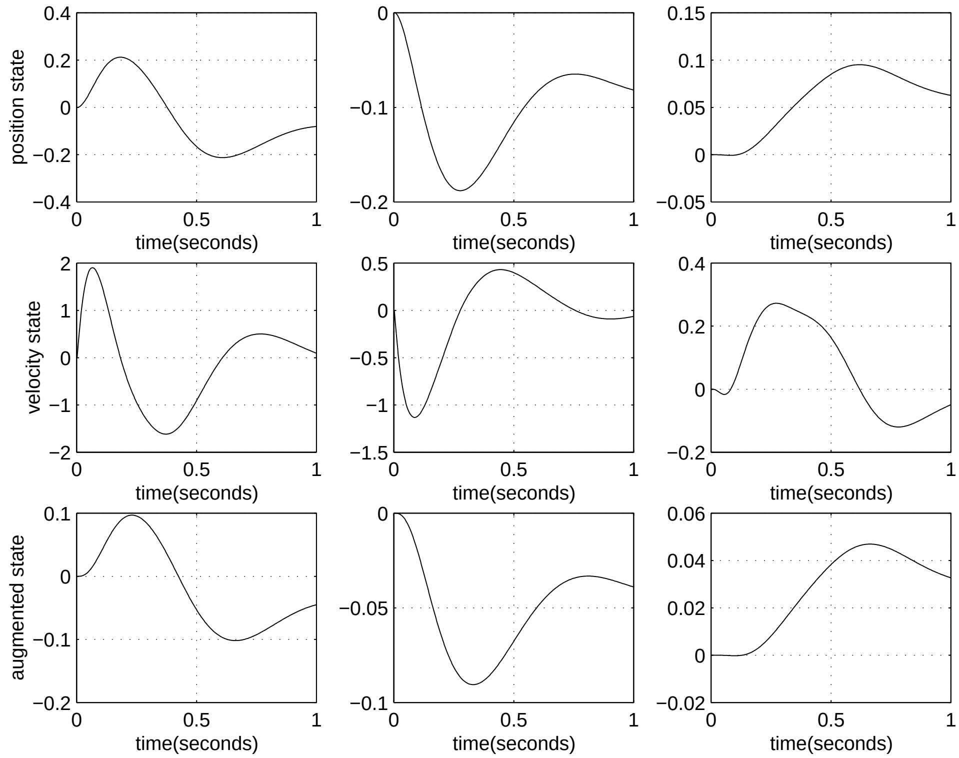

Figure 8: The nine state trajectories as functions of time for the translational motion of body 1 in simulation 3.

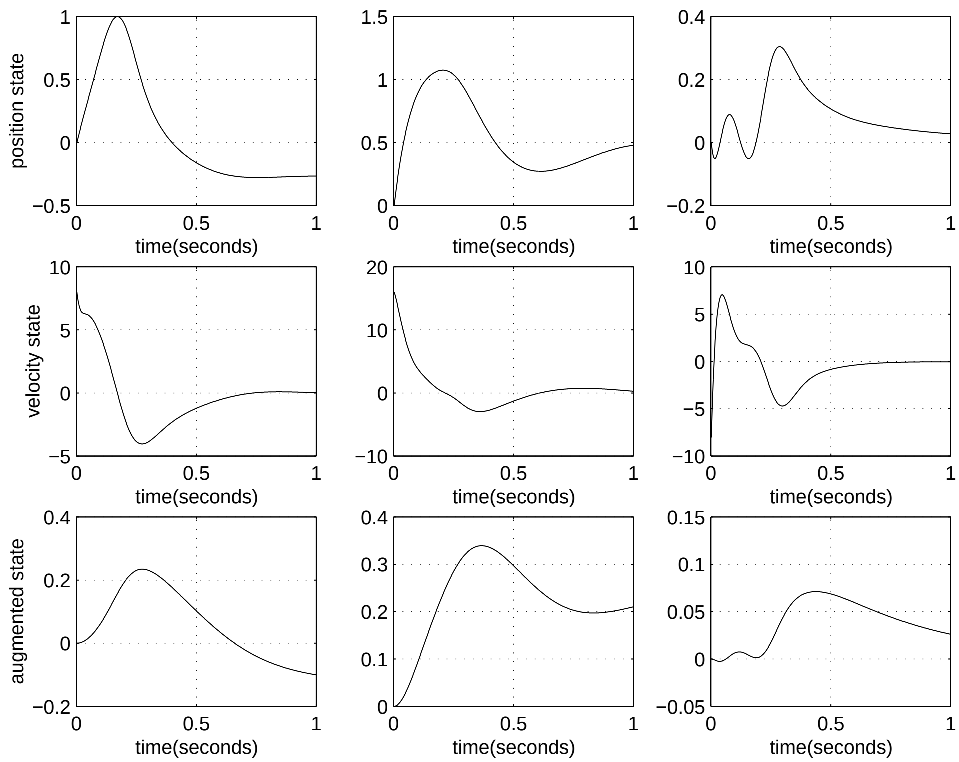

Figure 9: The nine state trajectories as functions of time for the rotational motion of body 2 in simulation 3.

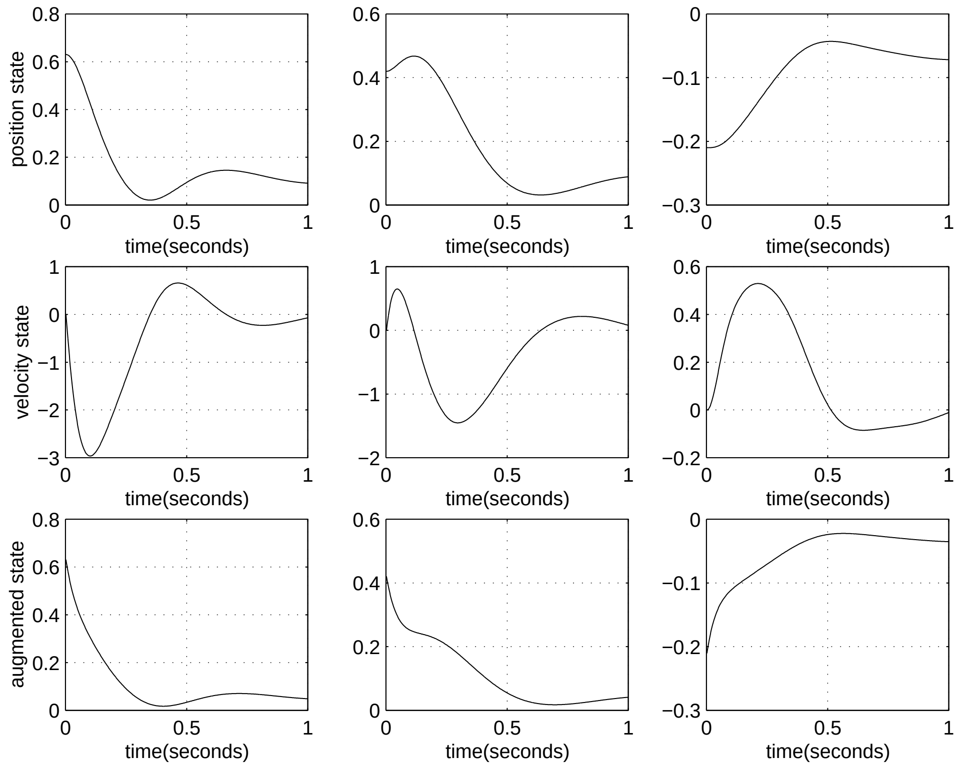

Figure 10: The nine state trajectories as functions of time for the translational motion of body 2 in simulation 3.

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