On the Dynamics and Lyapunov Stability of Constrained and Embedded Rigid Bodies

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Abstract

Lyapunov stability of constrained and embedded rigid bodies is considered. The constraints are of the equality type where the desired motion is to take place on an a priori defined sub manifold of movement. Special and augmented state spaces for the representation of systems of rigid bodies are presented. A systematic method of stabilizing these augmented systems and a procedure for constructing Lyapunov functions are presented. The representation is applicable to augmented as well as reduced state spaces of the system defined by the constraints. The augmented state space results from the embedding of the free rigid body system in the larger state space of free rigid body and position control states, and in which the Lyapunov function is constructed. The reduced state space results when the system is restricted and is reduced to lie on the sub manifold of movement.

It is shown that, for the class of rigid bodies and the physical constraints considered, the projected feedback structures, and the reduced Lyapunov function constitute appropriate stabilizing structures for the constrained system. It is shown that the method applies equally to holonomically constrained and visco-elastically coupled rigid bodies.

Digital computer simulations of one single rigid body system are presented to demonstrate the feasibility and effectiveness of the method. Applications to natural systems and the role of cartilage, ligaments, and muscles in maintaining the integrity and stability of the joints are noted.

keywords: Nonlinear analysis and design, constrained systems, rigid body dynamics, augmented state space, Lyapunov stability, global stability, natural joints.

1 Introduction

Systems of connected rigid bodies with holonomic, non holonomic and soft constraints [1–3] can adequately model robotic, humanoid, and locomotion systems. Almost all of the modeling, computations, simulations, internal modeling, and animation of such systems are carried out by computers. It is imperative to have state space representations that are symbolically rich and efficient, and relegate as much of the details of formulae and manipulation to the computer as possible. Economy, precision and modularity of such representations are additional desirable attributes. One such representation is explored in this paper for a single rigid body. It is equally imperative to develop systematic nonlinear control methods for stability of single and multiple rigid bodies. One approach, based on Lyapunov stability, is partially explored in this paper for global stability of a free rigid body and a holonomically constrained single rigid body.

Further, all interactions of humans and robots with objects in their environment and all locomotion can be represented by making, breaking, and controlling contact constraints. Therefore, the issues of making, breaking, and maintaining constraints, while at the same time guaranteeing stability of the system, are important and relevant control issues. However, the problem of simultaneous maintenance of constraints and rendering the system globally stable is not pursued here.

The emphasis of the paper is on Lyapunov stability of permanently constrained three-dimensional rigid bodies. A procedure is developed for global stability of such systems in the range of system states and parameters where the Lipschitz condition is satisfied. In its simplest form, for the rigid body systems considered here, the satisfaction of the Lipschitz condition requires imposition of certain inequality constraints on the states. Thus, the imposition of the inequality constraints is an integral part of the overall design and study. The inequality constraints can be maintained in a rigid body system by one of several techniques:

1. Inclusion of hard constraints in the system to limit the range of the states [3];

2. Inclusion of sufficient stiffness to prevent the system from evolving close to the forbidden boundaries [4];or

3. Invoking intermittent repelling forces that guide the system away from the boundaries by the techniques of sliding mode [5], convex optimization [6], etc.

The issue of maintaining inequality constraints is not further pursued here. A combination of methods 1 and 3 above could be easily utilized here to maintain the required inequality constraints.

The emphasis, in this paper, is on equality constraints. Deriving robust stabilizing feedback and Lyapunov functions are the focus here. The procedure outlined, allows for the construction of Lyapunov functions in a systematic way, analogous to the methods of construction of such functions for linear stable time-invariant systems [7]. The procedure is demonstrated for one rigid body which is, subsequently, constrained.

The basic tenet of the approach is to embed the rigid body in a larger state space with certain inherent feedback structures and to use the total energy of the system as a Lyapunov function [8]. The starting point is to choose two suitable state spaces. The first is for translation in an inertial coordinate system (ics) where Cartesian positions and velocities constitute six states. The second state space is for rotation where Euler angles and the angular velocity of the body in its own principal axes coordinate system (bcs) are the states [9, 10]. Three augmented states are introduced for control of the constraint. More about the applications of augmented state space to natural and man made systems can be found in [11].

Equality constraints are sometimes part of the structure of a dynamic system, and, in that case, the equations of the system do not explicitly include the forces of constraint. This situation occurs in linear electrical circuits where there are closed meshes of capacitors and sets of inductors that converge to a node. Analog mechanical systems show the same behavior. In many other systems, the forces explicitly appear [1, 12], and the control of the magnitude of these forces, in addition to stability and trajectory control, is part of the overall design of the system.

Two major attributes of the approach presented here are:

1. The design is robust, and the system is relatively insensitive to parameter changes.

2. For the systems considered here, the Lyapunov function and the control structure for the free body can be projected on the sub manifold of movement, and are the corresponding Lyapunov function and the control strategy for the constrained system. The dynamics of a free rigid body are formulated first. The body is constrained; namely, the translation degrees of freedom are inhibited by fixing an arbitrary point of the body in the inertial coordinate system (ics). Several methods for the imposition of the constraint are discussed: sliding mode method [5, 13], nonlinear passive visco-elastic coupling that maintains the constraint, and an augmented nonlinear visco-elastic structure that also embodies active and adaptive elements. The latter structure can be used to model muscles and tendons [14, 15] that anticipate perturbation. One can program muscles to coactivate [16, 17] for the purposes of impedance control or enhancement of the forces that maintain joint constraints. Simulations of one rigid body are carried out to demonstrate the application of the approach and to demonstrate its feasibility.

2 Unconstrained Single Rigid Body

Let Θ and Ω be, respectively, the Euler angles and the angular velocity vector of the body expressed in the previously defined body coordinate system (bcs). Let X and V be the translational vectors of position and velocity of the center of gravity of the body, expressed in the inertial coordinate system (ics) system. Let Λ be a 3-vector of force acting at a point C on the body whose coordinates are vector R in bcs, figure 1. The force Λ may be a connection force to another body, a constraint force, a visco-elastic interaction force with the environment, a stabilizing force, a gravity compensating force or a random disturbance. With reference to vector R, we define the skew symmetric 3×3 matrix $\breve{\mathcal{R}}$ [3]. The vectors of force G and H are, respectively, the gravity vector and an equivalent or resultant vector of all forces [18] acting on the rigid body. Similarly N is an equivalent or resultant couple of all forces acting on the rigid body. Based on these simplifications, the equations of motion of the single rigid body are [9]:

$$
\begin{aligned}
\dot{\Theta} &= B(\Theta)\Omega \\
J\dot{\Omega} &= f(\Omega) + N + \breve{\mathcal{R}}A'\Lambda \\
\dot{X} &= V \\
M\dot{V} &= G + H + \Lambda
\end{aligned}
\tag{1}
$$

The matrices $A(\Theta)$, $B(\Theta)$ and $\breve{\mathcal{R}}$ are given below.
Let $A_1(\theta_1)$, $A_2(\theta_2)$ and $A_3(\theta_3)$ be defined by:

$$
A_1(\theta_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}.
$$

$$
A_2(\theta_2) = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}.
$$

$$
A_3(\theta_3) = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

Now $A(\Theta)$ can be defined:

$$
A(\Theta) = A_1(\theta_1)A_2(\theta_2)A_3(\theta_3) \tag{2}
$$

The matrix $B(\Theta)$ is given by:

$$
B(\Theta) = \begin{bmatrix} \frac{cos\theta_3}{cos\theta_2} & \frac{-sin\theta_3}{cos\theta_2} & 0 \\ sin\theta_3 & cos\theta_3 & 0 \\ \frac{-sin\theta_2 cos\theta_3}{cos\theta_2} & \frac{sin\theta_2 sin\theta_3}{cos\theta_2} & 1 \end{bmatrix}.
$$

It is to be observed that, matrix B is of finite norm only for a limited range of θ_2, i.e., θ_2 must lie in the open interval: $-\pi/2$ and $+\pi/2$. This is the range for which the Lipschitz's condition holds. For $\theta 1$ and θ_3, respectively corresponding to roll and yaw, the range is from $-\pi$ to $+\pi$ - corresponding to the range for the physical three-dimensional world. All the development and results in this paper are limited to these ranges.

Let vector R have components r_1, r_2, and r_3. The skew symmetric matrix $\breve{\mathcal{R}}$ is defined as:

$$\breve{\mathcal{R}} = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}.$$

The ortho-normal transformation A defines the transformation of a vector, given in the bcs to a vector in the ics. . The inverse of A is A'.

For convenience, let us define the twelve-dimensional vector Z to represent the state of the rigid body where

$$Z' = [\Theta', \Omega', X', V']$$

The above twelve equations can be symbolically written as

$$(\mathcal{J})\dot{Z} = \mathcal{F}(Z, \Lambda, G, H, N) \tag{3}$$

where $\mathcal{J}$ is the diagonal 12×12 matrix whose diagonal elements are, respectively, from the four diagonal matrices I, J, I and M. Here I is the 3×3 identity matrix and M is the identity I multiplied by the mass m of the rigid body. It is sometimes desirable to use the coordinates of other arbitrary points on the surface rather than those of the center of gravity. This is due to ease of measurement and instrumentation. Let the arbitrary point be described by vector R in bcs and E in the ics. This is the point where the external force Λ acts on the body. Let the state of the rigid body in the new coordinate system be W:

$$W' = [\Theta', \Omega', E', F']$$

The coordinate transformation between W and Z is

$$\begin{aligned} \Theta &= \Theta \\ \Omega &= \Omega \\ E &= X + AR - R \\ F = \dot{E} &= V - A\breve{\mathcal{R}}\Omega \end{aligned} \tag{4}$$

Assuming that the ranges of the Euler angles are limited, respectively, as follows:

$$\begin{aligned} -\pi \leq \theta_1 &\leq \pi, \\ -\pi/2 \leq \theta_2 &\leq \pi/2, \\ -\pi \leq \theta_3 &\leq \pi, \end{aligned} \tag{5}$$

it can be shown that the transformation from the Z space to the W space is one to one and onto, and is given by:

$$\partial Z/\partial W \quad = \quad \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ A\breve{\mathcal{R}}B^{-1} & 0 & I & 0 \\ -A(\breve{\mathcal{R}}\Omega)B^{-1} & A\breve{\mathcal{R}} & 0 & I \end{bmatrix}. \tag{6}$$

The inverse of the matrix above is given by:

$$\partial W/\partial Z \quad = \quad \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ -A\breve{\mathcal{R}}B^{-1} & 0 & I & 0 \\ A(\breve{\mathcal{R}}\Omega)B^{-1} & -A\breve{\mathcal{R}} & 0 & I \end{bmatrix}. \tag{7}$$

With the above inverse, the equations of the rigid body in the W coordinate system are given by:

$$\dot{W} = \partial W/\partial Z(\mathcal{J}^{-1})(\mathcal{F}(Z(W),\Lambda),G,H,N) \tag{8}$$

The latter equation describes the dynamics of the single rigid body about a moving point. These equations should complement those given by Whittaker [19] where the equations of motion about a fixed point are discussed.

3 Permanently Constrained Rigid Body

Suppose a point on the surface of the rigid body is permanently connected to a specific fixed point in the inertial coordinate system. This means the translation coordinates of the center of gravity are limited to three-dimensional motion afforded by rotation about the fixed point. The connection can be described by three holonomic constraints. The motion of the constrained system can either be described in the original state space by showing that it is restricted to a sub manifold, or projected onto a reduced state space. As stated before, we assume the constraints are permanent, cannot be violated, and their control is not an issue here. The variable Λ corresponds to the force of constraint in this section.

3.1 Sub Manifold of Motion

In this section, the constraint sub manifold is first derived in the original state space of Z. Let us rewrite equation 1 in the following form:

$$\mathcal{J}\dot{Z} = \mathcal{E}(\mathcal{Z},\mathcal{G},\mathcal{H},\mathcal{N}) + c(Z)\Lambda. \tag{9}$$

where

$$\mathcal{E} = [V', (G+H)', (B\Omega)', (f(\Omega)+N)']'$$

and H and N are, up to this point, undefined control inputs. Vector c is the 12×3 matrix given below:

$$c' = [0, -A(\Theta)(\breve{\mathcal{R}}), 0, I] \tag{10}$$

Let a point of the body(with coordinates R in bcs) be attached to a point with coordinates

$$E = 0$$

in the ics system. This geometrical constraint, and its first two derivatives with respect to time can be specified by the following three-vector equations:

$$X + A(\Theta)R - R = 0 \tag{11}$$

$$V - A(\Theta)(\breve{\mathcal{R}})\Omega = 0 \tag{12}$$

$$\dot{V} - A\breve{\mathcal{R}}\dot{\Omega} - A(\breve{\Omega})^2 R = 0 \tag{13}$$

The latter acceleration equation can be written more compactly as:

$$c'\dot{Z} - A(\breve{\Omega})^2 R = 0 \tag{14}$$

It is noticed that the vector c in equation 9 and the coefficient of $\dot{Z}$ in equation 14 are the same. This fact follows from the principle of virtual work, and is consistent with the approach of Whittaker [19]. The formulation here is, however, somewhat more general, and is a variation of those in references [1] and [9]. Now, following the procedure in [1], the force of constraint Λ can be computed from equations 9 and 14:

$$\Lambda = -[c'\mathcal{J}^{-1}c]^{-1}(c'\mathcal{J}^{-1}\mathcal{E} - A\breve{\Omega}^2 R) \tag{15}$$

As is apparent from equation 15, the force of constraint Λ is function of the states and inputs to the system.

In order to simplify the dynamics, one can make use of a state-dependent projection operator $\mathcal{P}$. Let this operator be defined by:

$$\mathcal{P} = c[c'\mathcal{J}^{-1}c]^{-1}c'\mathcal{J}^{-1}. \tag{16}$$

It is easy to prove that $\mathcal{P}$ is a projection operator. The individual columns of c are eigen-vectors of $\mathcal{P}$ with eigen-values of unity. Nine other vectors, orthogonal to $c'\mathcal{J}^{-1}$ are the remaining eigen-vectors of $\mathcal{P}$ with zero eigenvalues. With these definitions, the sub manifold of motion, on which the system evolves, is given by:

$$\mathcal{J}\dot{Z} = (I - \mathcal{P})\mathcal{E} - c(c'\mathcal{J}^{-1}c)^{-1}A(\breve{\Omega})^2 R. \tag{17}$$

where I is the identity operator of dimension 12. The above development is also useful in problems of constraint control where the imposition, maintenance, and violation of constraints are to be dealt with. However, as stated before, these objectives are not further pursued here.

3.2 Projected Constrained System

Assuming that the constraints are permanent and cannot be violated, the equations of the system can be derived by projection onto a reduced state space. For standard methods of either reducing the dimension of the system and eliminating all the forces of constraint, or leaving the system in the original state space and computing all the forces of constraint, or combining these two techniques, see references [2,3]. Here, an alternative method, based on the previous section, is presented that allows the extension of Lyapunov methods from the free rigid body to the constrained rigid body case.

Let point C be permanently fixed to the ics. This means:

$$E = \dot{E} = F = \dot{F} = 0. \tag{18}$$

The projection of equation 8 onto the Θ, Ω space, and the computation of Λ results in

$$\begin{aligned} \dot{\Theta} &= B(\Theta)\Omega \\ \dot{\Omega} &= (J)^{-1}(f(\Omega) + N + \check{\mathcal{R}}A'\Lambda). \end{aligned} \tag{19}$$

It also follows that

$$\Lambda = -G - H - mA(\check{\check{\mathcal{R}}\Omega})\Omega + mA\check{\mathcal{R}}\dot{\Omega} \tag{20}$$

In the above equation, the force of constraint Λ is given as a function of the state, the derivative of the state, and the control inputs G and H rather than strictly as a function of the state and control inputs. From the above three equations, in principle, Λ can be either computed or eliminated. When Λ is eliminated, the following equations result:

$$\begin{aligned} \dot{\Theta} &= B(\Theta)\Omega \\ (J - m(\check{\mathcal{R}})^2)\dot{\Omega} &= f(\Omega) + N - \check{\mathcal{R}}A'(G + H) - m\check{\mathcal{R}}(\check{\check{\mathcal{R}}\Omega})\Omega \end{aligned} \tag{21}$$

The above equations are the standard reduced equations of the system for rotation about an arbitrary fixed point. When the fixed point happens to lie on one of the principal axes, the equations simplify further.

4 Embedded Rigid Body

Many natural systems and man-made systems have skeletons that can be approximated by an interconnected rigid body system. A large number of muscles and actuators, with their own specified dynamics, control this rigid body system. It is desirable, in stability studies, to include the dynamics of passive and active controllers to that of the rigid body system. The passive controllers, in many natural systems, consist of cartilage and ligaments. The active controllers, in natural systems, are the totality of muscles. The inclusion of controller, actuator and sensor dynamics [?,8]in stability studies requires augmenting the state space of the rigid body system by the states of the additional dynamic components. In our single rigid body case, the state space of the rigid body could be embedded in a larger state space where the twelve states of the rigid body are augmented by the states of three visco-elastic connectors or controllers. In this paper the case of a three-state connector or controller is considered.

The free rigid body is represented by state space equation 1. The state is Z. The inputs are gravity G, the control inputs H and N, and the control force vector Λ. In this section Λ has a different interpretation than in the previous section. Here, Λ is a visco-elastic passive feedback force that is utilized primarily to maintain the constraint. The nonlinear feedback quantities H and N are primarily used for stabilization. The block diagram of the embedded system is shown in fig 2 The outputs of the rigid body system are E and F, taken from equation 4. The objective here is to stabilize the system and maintain the constraints:

$$E = 0$$

and

$$F = 0.$$

Let D be the state, E and F be the inputs, and Λ be the output of the controller. Let K_c, L_c ,K_d, and L_d be 3×3 positive definite symmetric matrices. The controller is defined by the following state and output equations:

$$\begin{aligned} \dot{D} &= (L_d)^{-1}(K_c(E-D) - K_d D) \\ \Lambda &= -K_c(E-D) - L_c F \end{aligned} \tag{22}$$

The block diagram of figure 2 shows the rigid body in box 1, the output construction in box 2, the stabilizing nonlinear state feedback in box 3 and the passive output feedback for maintaining the constraints in box 4.

The control structure of figure 2 can also be utilized to model physical active actuators such as muscles and tendons [14]. It has been shown in reference [17], and by others, that under coactivation of a pair of agonist and antagonist muscles, the pair, primarily, behaves as a bi-directional spring (or position feedback). However, the biomechanical implications and applications of the contoller in equation 22 are not discussed further here.

5 Control Structure and Stability

Consider the system given by equation 1. The state feedback mechanism for stability of the rotational degrees of freedom of the free body is discussed as follows [8]. Let K_r and L_r be two 3×3 positive definite, and symmetric matrices. The state feedback for control is given below,

$$N = -B'(\Theta)K_r\Theta - L_r\Omega \tag{23}$$

A similar control structure is proposed for the translational motion. Let K_t and L_t be two 3×3 positive definite and symmetric matrices. The control is given below:

$$H = -K_t X - L_t V. \tag{24}$$

The control structure above is effective and robust. Lyapunov methods can be used to establish stability of the system in the range of rotational states of equation 5. Let v be a candidate Lyapunov function that physically corresponds to the stored energy of the system - kinetic, elastic, and the potential energy - plus a term ϵ that corresponds to the energy transfered from the force Λ to the system. The translational and rotational energies, embodying the kinetic and elastic ones, are, respectively, given by:

$$\begin{aligned} v_t &= 0.5V'MV + 0.5(X)'K_t(X) \\ v_r &= 0.5\Omega'J\Omega + 0.5(\Theta)'K_r(\Theta). \end{aligned} \tag{25}$$

The energy transferred to the system by the force Λ is as follows

$$\begin{aligned} \dot{\epsilon} &= (\Omega'\breve{\mathcal{R}}A' + V')\Lambda \\ \epsilon &= \int_0^t (\Omega'\breve{\mathcal{R}}A' + V')\Lambda dt. \end{aligned} \tag{26}$$

The potential energy p is given by:

$$p = -\int_0^t V'G dt. \tag{27}$$

Because of the two integrals, an arbitrary constant h may be involved in the final form of the Lyapunov function. Alternatively, this issue of a constant can be argued based on physical conditions. Since the origin for the potential energy, i.e., where the potential energy is assumed to be zero, is somewhat arbitrary, the Lyapunov function is defined here within a constant.

The overall Lyapunov function candidate v is, therefore given by

$$v = v_t + v_r - \epsilon + p + h \tag{28}$$

The above candidate Lyapunov function is very general, and its positivity and value of zero at the equilibrium point have to be established now. With regard to ϵ, when Λ is the holonomic force of permanent connection, the force does not do any work and the integral is zero. When Λ is visco-elastic, state-dependent or of a more complex form, the analytical manipulation of the integral is often difficult, if not impossible, and computational and numerical methods are indispensable. The value of $v = 0$ at the equilibrium point can almost always be set by choice of the constant h. For the positivity of v, several cases are considered later. Now consider the system of equation 1 with the nonlinear feedback of equations 23 and 24. Differentiating v with respect to time results in the following negative semi-definite quantity:

$$\dot{v} = -V' L_t V - \Omega' L_r \Omega. \tag{29}$$

Since $\dot{v}$ is not identically zero along any trajectories of the system, one can invoke La Salle's theorem [20] to prove that the system is asymptotically stable.

5.1 Free Rigid Body

We consider the case of

$$\Lambda = 0.$$

Due to gravity, the system settles at an equilibrium point X_e given by

$$X_e = {K_t}^{-1} G.$$

In order for the Lyapunov function to be zero at this equilibrium point, $h = 0.5G'{K_t}^{-1}G$. With the help of an auxiliary quantity $K_a = {K_t}^{0.5}$ the Lyapunov function v can be written as a quadratic quantity in the state, and therefore it is positive.

$$v = 0.5(K_a X - {K_a}^{-1} G)'(K_a X - {K_a}^{-1} G) + .5V' M V + 0.5\Omega' J \Omega + 0.5(\Theta)' K_r(\Theta). \tag{30}$$

It is to be noted that

$$(K_a X - K - a^{-1} G)'(K_a X - {K_a}^{-1} G) = (X - X_e)' K_t (X - X_e).$$

It can be argued that the above control structure is robust to system parameter changes. The elastic and viscous gains can be selected large enough to account for varaiations in the system parameters.

5.2 Holonomically Constrained Rigid Body

Suppose that the free rigid body is constrained such that the point where force Λ is applied is permanently connected to the ics coordinate system. Let us assume Λ to be the force of constraint and not a disturbance. Consequently, the constraints remain satisfied throughout the evolution of the system in time, and the degrees of freedom of the system are reduced by three. This implies that the imposing and bringing about the constraints is not part of the control problem. The objective here is to show that the same stabilizing feedback structure and the same Lyapunov function, respectively, stabilize, and are applicable to the constrained system. We assume that the equilibrium points of the free body system and the constrained system are to coincide. It is proven below that the same energy Lyapunov function that was established for the free body system is a proper Lyapunov function for the constrained system.

Let a Lyapunov function v be constructed from equations 25 and 27:

$$v = v_t + v_r + p. \tag{31}$$

The projection from the 12-dimensional Z space of the free body to the six - dimensional space of Θ and Ω is given by

$$\begin{aligned} \Theta &= \Theta \\ \Omega &= \Omega \\ X &= -A(\Theta)R + R + X_e \\ V &= A(\Theta)(\breve{\mathcal{R}})\Omega. \end{aligned} \tag{32}$$

The nonlinear feedback controls result from substitution of equation 32 in equations 23 and 24:

$$\begin{aligned} N &= -B'(\Theta)K_r\Theta - L_r\Omega \\ H &= -K_t(-A(\Theta)R + R + X_e) - L_tA\breve{\mathcal{R}}\Omega. \end{aligned} \tag{33}$$

The projected Lyapunov function is

$$v_c = 0.5(R - AR)'K_t(R - AR) + .5(A\breve{\mathcal{R}}\Omega)'M(A\breve{\mathcal{R}}\Omega) + 0.5\Omega'J\Omega + 0.5(\Theta)'K_r(\Theta). \tag{34}$$

We note that

$$v_c(0) = 0$$

and that, otherwise,

$$v_c > 0$$

in view of its quadratic form.

Now, the equations of the reduced system are developed in some detail. Substitution of equation 32 in equation 1 and multiplying both sides by the 3×12 matrix $[0, I, 0, \breve{\mathcal{R}}A]$ results in the elimination of the constraint force Λ from the equations of the projected system:

$$(J - \breve{\mathcal{R}}m\breve{\mathcal{R}})\dot{\Omega} = f(\Omega) + N + \breve{\mathcal{R}}m(\breve{\Omega})^2(R) - \breve{\mathcal{R}}A(G + H). \tag{35}$$

It is to be noted that equation 35 is the equation of an inverted pendulum fixed at a point with body coordinate R to the inertial coordinate system at point X_e. Suppose the body's coordinate system is translated to this point. Let J_n and f_n define the moment of inertia J and the corresponding function f relative to a translated principal axes- based body coordinated system. Then, equation 35 simplifies to the following form:

$$J_n\dot{\Omega} = f_n(\Omega) + N + \breve{\mathcal{R}}A(K_t(R - AR) + L_tA\breve{\mathcal{R}}\Omega). \tag{36}$$

Equation 36 and

$$\dot{\Theta} = B\Omega$$

are the state space equations of the projected(reduced) system. It can be shown that the derivative of the projected Lyapunov function, i.e., equation 34 is

$$\dot{v_c} = -(\Omega)' L_r \Omega + \Omega' \breve{\mathcal{R}} A' L_t A \breve{\mathcal{R}} \Omega. \tag{37}$$

Therefore $\dot{v_c}$ is nonpositive, and, by invoking La Salle's theorem, the constrained system is globally stable. An alternative, but more intuitive proof is to first transform the 12-dimensional Z system to a modified W system that accounts for gravity bias:

$$\begin{aligned} \Theta &= \Theta \\ \Omega &= \Omega \\ E &= X + AR - R - X_e \\ F &= \dot{E} = V - A\breve{\mathcal{R}}\Omega. \end{aligned} \tag{38}$$

Now in the new coordinate system, one restricts the controls, the system and the Lyapunov function to the subspace of Θ and Ω.

5.3 Imposition of Constraints

A more frequently occurring situation in robotics is the requirement to solve the simultaneous problems of the stability and maintenance of constraints. When the rigid body is not rigidly connected to the ics system, additional control is needed in order to maintain the constraints. In such a situation, the system may be stable with the above positive definite structures of equations 23 and 24, but the system may chatter in and out of the sub manifold of movement. In order to reduce the excursions of the system outside the manifold of motion, the concept of sliding mode control [5] can be applied to provide a solution to the control problem. This method amounts to selecting a control as follows:

$$\Lambda = -K_s sgn(E), \tag{39}$$

where K_s is a large constant. The system may still chatter in and out of the sub manifold of movement. A second method is to assume Λ to be a large visco-elastic force that imposes the constraints and keeps the rigid body motion on the manifold of motion. Let the distance of the contact point on the body be at a distance C from the contact point in the ics system, i.e., vector E of equation 4 and its derivative with respect to time F. Let the visco-elastic sliding force be given by:

$$\Lambda = -K_c E - L_c F. \tag{40}$$

The additional controls N and H, needed for stability, are taken to be the same as in equations 23 and 24. The block diagram of the system is given in figure 2. Let v_e be the elastic energy of the sliding controller of equation 40:

$$v_e = 0.5 \;\; E' K_c \;\; E. \tag{41}$$

The system is given by equation 1. For simplicity, we set

$$G = 0.$$

Let the Lyapunov function be given by

$$v = v_t + v_r + v_e \tag{42}$$

It can be shown that

$$\dot{v} = -V' L_t V - \Omega' L_r \Omega - F' L_c F. \tag{43}$$

Therefore

$$\dot{v}, \leq 0$$

and by invoking La Salle's theorem, the system is globally stable. Although the constraints may not always be identically satisfied, they are driven towards zero. An example of the performance of the system is shown in Case 2 of the next section by numerical simulation.

In many practical systems, the controllers, sensors and actuators may have dynamics of their own, and the stability of the overall system is one of the main objectives of control. The methodology presented here can be extended to larger systems. A simple case is presented here. For simplicity, again we assume the gravity is zero, and the controllers H and N are the same as before. Let the controller for imposing the constraints be given by equation 22. The objective here is to compare the performane of the latter strategy to that of equation 40. The global stability of the embedded system is shown below.

Let v_i be the elastic energy of the controller:

$$v_i = .5(E - D)' K_c (E - D) + .5 D' K_d D. \tag{44}$$

Consider a Lyapunov function consisting of the sum of the kinetic and elastic energies:

$$v = v_t + v_r + v_i. \tag{45}$$

The time derivative of the Lyapunov function is

$$\dot{v} = -V' L_t V - \Omega' L_r \Omega - F' L_c F - \dot{D}' L_d \dot{D}. \tag{46}$$

It can be seen that the embedded system is globally stable, where again La Salle's theorm is invoked.

6 Simulation Results

The results of simulating the transient behavior of one constrained rigid body in three cases are briefly reported in this section. All the numerical parameters and gains are listed in the Appendix, and taken from reference [21]. Simulation 1 displays the behavior of the Permanently constrained rigid body with the representation of equation 21. Simulation 2 considers the same single rigid body where the constraints are imposed by the visco-elastic control, equation 40. Simulation 3 considers the system with augmented control of equation 22 for maintaining the constraints. The ability to systematically derive controllers and guarantee stability for constrained rigid body systems is of theoretical as well as practical importance. From a theoretical point of view, this approach allows one to articulate nonlinear feedback structures that stabilize large-dimensional rigid body systems that arise in robotics, humanoid studies, and living systems where a skeleton may be modeled by a constrained rigid body system. The development is analogous to stability of linear systems and derivation of appropriate Lyapunov functions [7]. From a practical point of view, the approach allows one to implement nonlinear stabilizing feedback with the help of on-line computation. The only drawback is that the system actuators may not be able to deliver the necessary large gains [15] due to their physical and physiological limitations and saturation.

A second reason for the need for simulation and for utilizing computational methods is that the disturbance input may have involved and complex dependence on the input and the state. In such a case, ascertaining positivity of the more general Lyapunov function, i.e., equation 28, and negativity of its time-derivative may only be feasible by computational methods and simulations.

Three simulations of the transient behavior of one rigid body are briefly presented in this section.

Simulation 1 displays the behavior of the permanently constrained rigid body with the representation of equation 21. Simulation 2 considers the same single rigid body where the constraints are imposed by the visco-elastic control, equation 40. Simulation 3 considers the system with augmented control of equation 22 for maintaining the constraints.

6.1 Case 1, Permanently Constrained System

In this simulation, the system is represented by equation 21. The initial state is

$$[0, 0, 0, 16, 32, -16]$$

The control is specified by projection of the translational controls:

$$X = -AR + R$$

and

$$V = A\breve{\mathcal{R}}\Omega$$

where

$$H = -K_t X - L_t V$$

and

$$N = -B' K_r \Theta - L_r \Omega.$$

The Lyapunov function is given by equation 31. The simulation time is two seconds. The trajectories of the states are given in figure 3. The final state at $t = 2.00$ is

$$[0.092, -0.1221, 0.0003, 0.0359, 0.0215, -0.0099].$$

6.2 Case 2, Imposition of Constraint

In this simulation, the system of equation 1 is simulated. The initial state is

$$X = V = \Theta = 0$$

and

$$\Omega = [16, 32, -16]'.$$

The controls H and N are from equations 23 and 24, while Λ is from equation 40. The visco-elastic connectors here could correspond to the action of cartilage and ligaments in living systems, and these functions could further be enhanced by coactivation of muscles [16]. The Lyapunov function is from equation 41. The system is simulated for two seconds with the same initial conditions as in case 1. The rotational state trajectories, and the Lyapunov function are shown in figure 4. The translational states and the constraints are shown in figure 5. It can be seen that, at the beginning of motion, all three constraints are violated by about -5, 2 and 2 centimeters. The final state after two seconds is

$$[X', V'] = [0.0015, -0.0024, -0.004, -0.100, 0.0007, 0.0000]$$

$$[\Theta', \Omega'] = [0.0057, 0.0035, 0.0000, -0.0014, -0.0240, 0.0001]$$

This means the system is stable, and the constraints are maintained.

6.3 Case 3, Embedded Constrained Rigid Body

A passive augmented state controller was simulated. The system is given by equation 1. The control is given by equations 22, 23 and 24.

The initial state is the same as in case 2. The results of this simulation show relative improvement in the maximum values of the excursion of the constraints from their desired values, i.e.:

$$E = 0.$$

The rotational state trajectories and the Lyapunov function and its derivative, with respect to time, are shown in figure 6. The translational state trajectories and the constraints are plotted in figure 7. The simulation establishes the better performance of the system with more complex visco-elastic control than Case 2.

7 Discussion and Conclusions

It has been shown that passive visco-elastic forces and couples, corresponding to nonlinear position and velocity feedback can globally, in the range of Lipschitz condition, stabilize a rigid body. Further, it has been shown that the energy of the rigid body, composed of the sum of the elastic, kinetic and potential, and the work done by external control or disturbance forces and couples, can be selected as a Lyapunov function.

More importantly, it was shown that the same control and Lyapunov function are applicable to a rigid body of reduced dimension, brought about by mechanical constraints. Both the control and the Lyapunov function are, in this case, projected onto the sub manifold of movement. It was also shown that the approach can be extended to a rigid body that is embedded in a larger state space due to augmented control states.

This methodology should make the design of stabilizing feedbacks and applications of Lyapunov stability more systematic and numerically more tractable for rigid body systems. With on-line computers, this method should find practical applications as far as the design of nonlinear functions is concerned. It should also find applications in mechatronics, large system design, and system integration where an overall system is composed of rigid bodies, controllers, computational units, and sensors. Finally, the modularity of the rigid body systems, their actuation, the generality of the feedback structures and the candidate Lyapunov functions should allow application to large dimensional systems in animation, robotics, humanoid studies, and modeling of living systems.

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9 Appendix

The definitions, symbols, and numerical values for a single rigid body system [3, 21] are given in Table 1. The one rigid body parameters used in the simulation of case 1 are:
$R = [0.15, 0.10, -0.42]'$,

$K_t = Diag(4100, 4100, 3280)$
$L_t = Diag(210, 210, 140)$
$K_r = Diag(400, 400, 320)$
$L_r = Diag(60, 60, 42)$

The parameters in the simulation of case 2 are:
$K_t = Diag(4100, 4100, 3280)$
$L_t = Diag(200, 200, 140)$

The additional parameters for simulation of case 3 are:
$K_c = Diag(12300, 12300, 9840)$
$L_c = Diag(600, 600, 420)$
$K_d = K_c$
$L_d = 2L_c$

Table 1: Definition, symbols, and numerical values for one rigid body.

	symbol	value	unit
mass.	m	41.00	Kg
p.m. of i.	j1	10.0	Kg m^2
p.m. of i.	j2	8.0	Kg m^2
p.m. of i.	j3	0.4	Kg m^2
c. of g.	l	0.42	m
gravity	g	10.0	m/s^2

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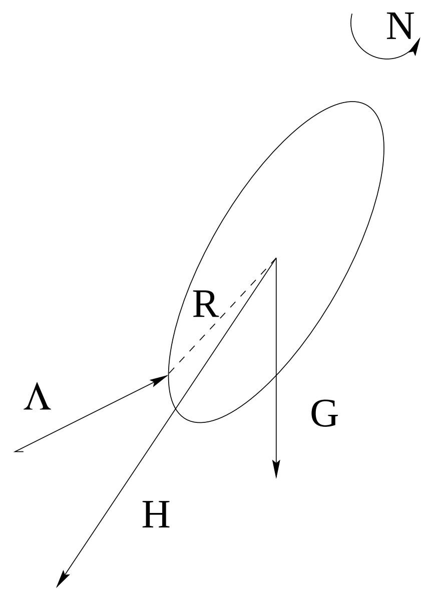

Figure 1: The free rigid body with gravity input G, force and couple inputs H and N, and external force Λ applied at point C with coordinates R in bcs.

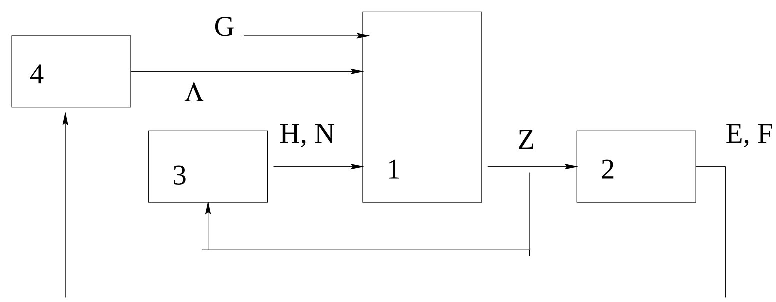

Figure 2: The block diagram of the embedded rigid body for stabilizing and maintaining constraints. 1: Dynamics given by equation 1; 2: Output equations given by equation 4; 3: Stabilizing state feedback given by equations 23 and 24; and 4: Controller given by equation 22.

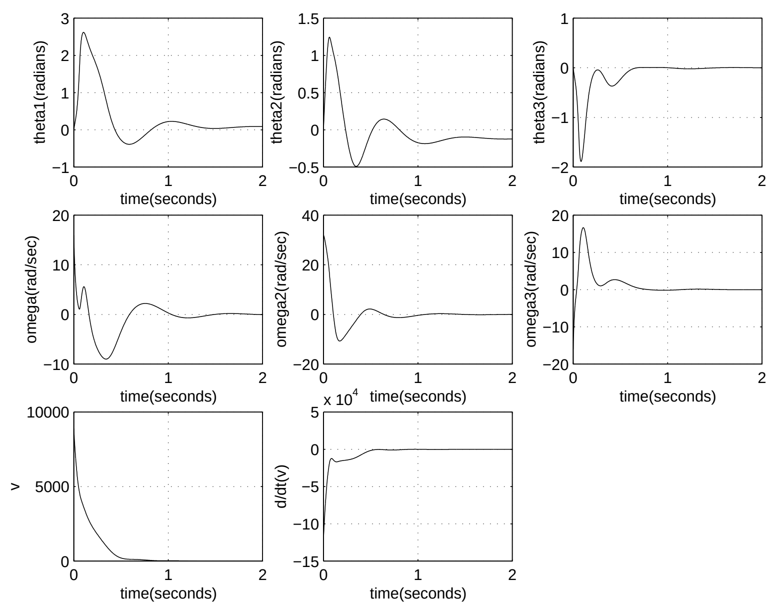

Figure 3: The state trajectories and the Lyapunov function and its derivative as functions of time for Case 1.

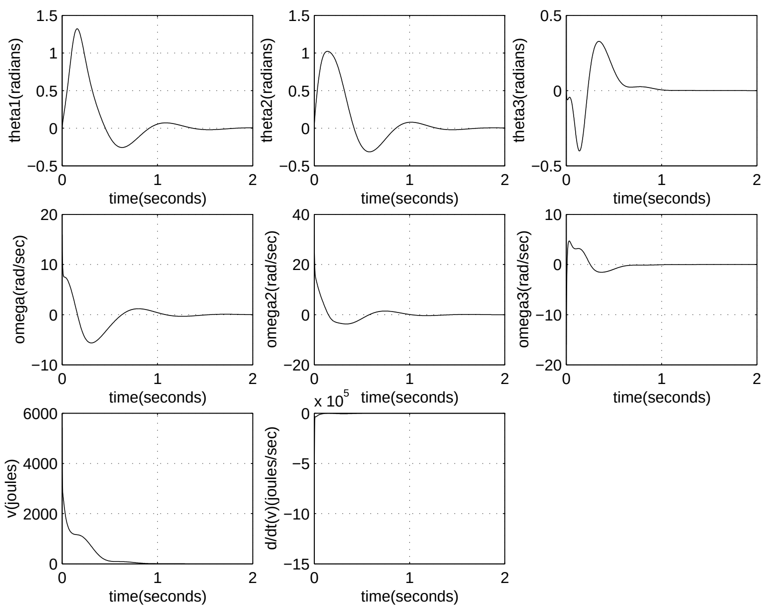

Figure 4: The rotational state trajectories and the Lyapunov function and its derivative for Case 2 plotted as functions of time for two seconds.

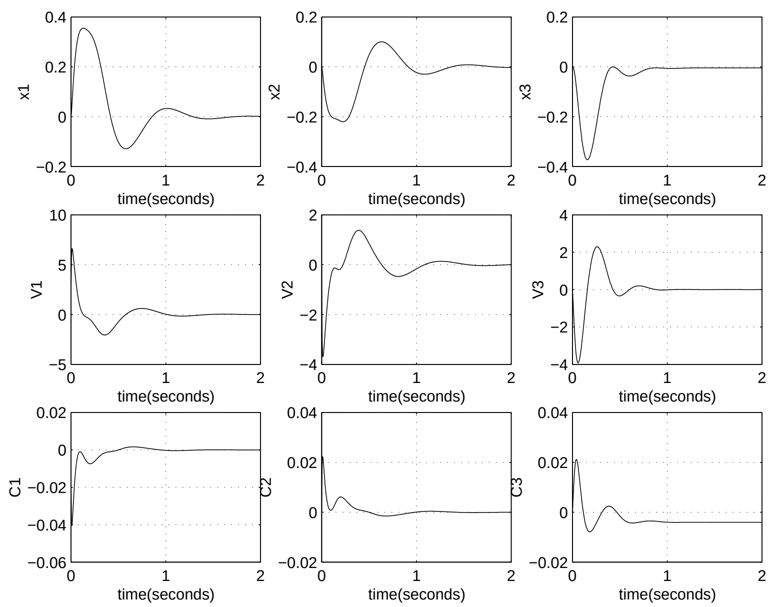

Figure 5: The translational state trajectories and the constraint E as functions of time for Case 2. Vector $C = E$ designates the constraint.

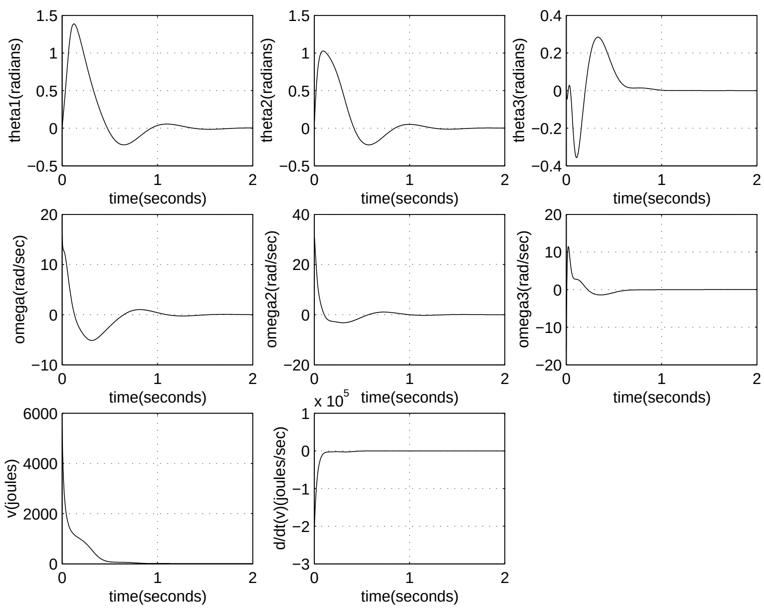

Figure 6: The rotational state trajectories and the Lyapunov function and its derivative for Case 3 plotted as functions of time for two seconds.

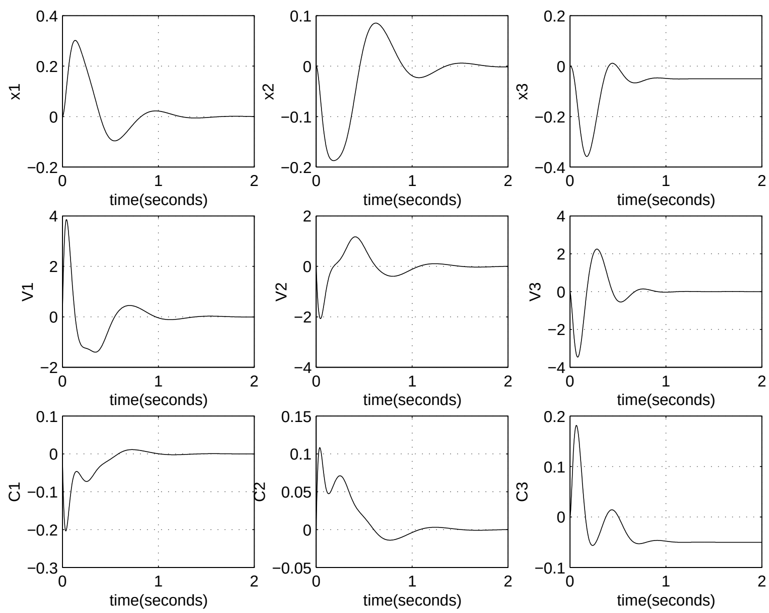

Figure 7: The translational state trajectories and the constraint E as functions of time for Case 3. Vector $C = E$ designates the constraint.