

PROBLEM SET 1
ECE 816

Assigned: March 30th
Quiz: Week 3

Instructor: Joel Johnson

Problem 1 (note page 5 of this document contains useful integrals for this problem)

(a) Show that the expected value and variance of a Rayleigh distributed random variable are $a\sqrt{\frac{\pi}{2}}$ and $\frac{\pi}{2}a^2\left(\frac{4}{\pi} - 1\right)$ respectively.

(b) Find the characteristic function of an exponentially distributed random variable with mean w_0 .

(c) Consider two random variables X and Y which have joint probability density function

$$f_{X,Y}(x,y) = \frac{1}{2\pi ab\sqrt{1-c^2}} e^{-\left[\frac{1}{2(1-c^2)}\left(\frac{x^2}{a^2} - 2\frac{c}{ab}xy + \frac{y^2}{b^2}\right)\right]}. \text{ Find } \langle X \rangle, \langle Y^2 \rangle, \text{ and } \langle XY \rangle. \text{ Are } X \text{ and } Y$$

correlated? Are they independent?

(d) Consider two random variables X and Y distributed according to the joint pdf of part (c) with $a = b$ and $c = 0$. Define these random variables to be the real and imaginary parts of a complex electric field. Find the pdf of the magnitude of this field, i.e. the pdf of a random variable $R = \sqrt{X^2 + Y^2}$.

Problem 2

A plane wave propagating in the \hat{x} direction in free space with a \hat{z} directed electric field and of frequency ω is incident upon a small lossy sphere of radius a and permittivity $\epsilon = 3 + i0.1$. The radius of the sphere is such that $ka \ll 1$.

(a) Write the backscattered electric field, scattering amplitude, and radar cross section (i) if the sphere is centered on the origin and (ii) if the sphere is located at position \vec{r}' . Note part (ii) will require a reconsideration of the Rayleigh scattering problem.

(b) Now consider a stochastic problem in which the position of the sphere is given by $\vec{r}' = \hat{x}b$, where b is a uniformly distributed random variable for $0 < b < b_0$. Find the expected value of the scattering amplitude in this problem as a function of b_0 . Interpret your results for large and small values of kb_0 .

(c) Find the expected value of the radar cross section in part (b). Interpret your results, and explain the relationship between your part (b) and (c) answers.

(d) Repeat parts (b) and (c) for b a zero mean Gaussian random variable with variance b_0^2 .

Problem 3

In this problem, consider Rayleigh scattering from two small spheres, each with polarizability α , but apply the “discrete dipole approximation” (DDA) to include coupling effects between the two spheres. See pages 3 and 4 on the DDA to formulate your solution.

(a) Consider the same configuration as in problem 2 part(a) except that now there are two spheres: one centered on the origin and a second at $\bar{r}_2 = \hat{x}d$. Find the backscattered scattering amplitude and radar cross section in the far field. An analytical answer should be obtained.

(b) Compare your answers to those obtained if the coupling between the spheres is neglected. When is a significant error introduced by neglecting coupling effects?

(c) Consider 1 GHz backscattering from two spheres in the configuration of part (a), each with $\epsilon = 3 + i0.1$ and radius 1 mm (note for a sphere $\alpha = \frac{3(\epsilon - 1)}{(\epsilon + 2)}\epsilon_0\left(\frac{4}{3}\pi a^3\right)$). Find the average scattering amplitude and radar cross section if d is assumed to be uniformly distributed for $0 < d < d_0$, both with and without coupling effects. You will need to compute these expected values numerically, trying several different values of d_0 . Discuss differences between the two as d_0 becomes large or small. Do your conclusions about error differ from those in part (b)?

(d) Discuss application of this technique as the number of scatterers becomes greater than 2. Would you expect to obtain an analytical solution? What would a numerical solution require?

Problem 4

Find the modified Stokes vectors for the following plane waves; note you will need to specify a coordinate system.

(a) $\bar{E} = (\hat{x} + 2\hat{y})e^{-ikz}$

(b) $\bar{E} = [\hat{y}(3 + i) - \hat{z}(2 - 2i)]e^{ikx}$

(c) $\bar{E} = (\hat{y} + i\hat{z})e^{ikx}$

(d) The sum of the fields in parts (b) and (c). Compare to your parts (b) and (c) answers.

The Discrete Dipole Approximation

Most of the scattering problems we have considered in previous courses have involved only one scattering object. When multiple scattering objects are present, obtaining analytical solutions becomes extremely difficult due to scattering interactions between the objects. In ECE 816 we will be considering approximate ways to address this problem, but we can also formulate equations which describe scattering with multiple objects “exactly” and attempt to solve them numerically. The DDA is a procedure for formulating multiple scatterer problems in terms of a matrix equation; essentially it is based on the same ideas as a point matching method of moments. However the DDA is somewhat simpler because it is assumed that all scattering objects are point dipoles. Such an approximation should only be reasonable when the objects are small in terms of the wavelength and when scattered fields are observed at distances greater than the size of the objects.

Since all of our scattering objects are point dipoles, the fields they radiate are determined completely if we know the dipole moment of each scatterer. Since the dipole moment is a vector, this means we have three scalar dipole moment components for each scatterer which are unknown. If the total number of scatterers is N , the total number of unknowns is therefore $3N$. The DDA formulates a matrix equation to determine these dipole moment components; once these are known total scattered fields can be determined anywhere in space.

Begin by expressing the total electric field at a point in space \vec{r}_j which is not on one of the scatterers as the sum of incident field and the scattered fields from each object:

$$\vec{E}(\vec{r}_j) = \vec{E}^{inc}(\vec{r}_j) + \sum_{k=1}^N \vec{E}_k^{dip}(\vec{r}_j) \quad (1)$$

where $\vec{E}_k^{dip}(\vec{r}_j)$ is the electric field radiated to point \vec{r}_j by the k th point dipole \vec{P}_k . Considering a point dipole in an arbitrary orientation and at an arbitrary location \vec{r}_k , the radiated field can be written as

$$\vec{E}_k^{dip}(\vec{r}_j) = -[A(\vec{r}_j, \vec{r}_k)] \cdot \vec{P}_k,$$

where the matrix $[A]$ is a 3 x 3 matrix whose elements A_{mn} can be written in dyadic form as

$$A_{mn} = \frac{\hat{a}_m}{4\pi\epsilon_0} \cdot \left[\frac{e^{ikr_{jk}}}{r_{jk}} \left(k^2 (\hat{r}_{jk}\hat{r}_{jk} - [I]) + \frac{ikr_{jk}-1}{r_{jk}^2} (3\hat{r}_{jk}\hat{r}_{jk} - [I]) \right) \right] \cdot \hat{a}_n.$$

The above equation may look confusing, but actually is simply expressing the radiated field of a point dipole in both near and far fields. In this equation, the unit vectors are

$$\hat{a}_1 = \hat{x}, \hat{a}_2 = \hat{y}, \hat{a}_3 = \hat{z},$$

and the vector from source to observation point is given by

$$\vec{r}_{jk} = \vec{r}_j - \vec{r}_k = r_{jk}(\hat{r}_{jk}),$$

with $r_{jk} = |\vec{r}_{jk}|$. Finally $[I] = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$ is the unit dyad. Dyadic notation contains terms involving products of unit vectors, for example $\hat{x}\hat{x}$. However these terms always appear with dot

products on the left and right, so that $\hat{a}_m \cdot (\hat{x}\hat{x}) \cdot \hat{a}_n = (\hat{a}_m \cdot \hat{x})(\hat{x} \cdot \hat{a}_n)$. Applying this rule throughout the equation should eliminate any confusion with the dyadic notation. You may want to compare this expression with typical point dipole fields for further clarification.

Fields evaluated on one of the point dipoles are written similarly, but the “self” radiation is dropped:

$$\bar{E}(\bar{r}_j) = \bar{E}^{inc}(\bar{r}_j) + \sum_{k=1, k \neq j}^N \bar{E}_k^{dip}(\bar{r}_j) = \bar{E}^{inc}(\bar{r}_j) - \sum_{k=1, k \neq j}^N [A(\bar{r}_j, \bar{r}_k)] \cdot \bar{P}_k$$

on point dipole j .

To complete the derivation, the relationship between the electric field exciting a sphere and induced dipole moment is written as $\bar{P}_j = \alpha_j \bar{E}(\bar{r}_j)$, where α_j is the “polarizability” of the j th small scatterer. Combining these equations together produces

$$\frac{\bar{P}_j}{\alpha_j} + \sum_{k=1, k \neq j}^N [A(\bar{r}_j, \bar{r}_k)] \cdot \bar{P}_k = \bar{E}^{inc}(\bar{r}_j)$$

which can be re-written as a standard linear system by combining the first term into the second to create a new matrix:

$$[M(\bar{r}_j, \bar{r}_k)] \cdot \bar{P}_k = \bar{E}^{inc}(\bar{r}_j)$$

Solving this matrix equation provides the \bar{P}_k 's; once known scattered fields at any point in space can be determined from equation (1). It is also possible to compute the total power absorbed by each scatterer.

See the following references for more information:

Draine, B. T. and Flatau, P., “The discrete-dipole approximation for scattering calculations,” J. Opt. Soc. Am. A, vol. 11, no. 4., pp. 1491-1499, 1994.

Flatau, P., “Improvements in the discrete-dipole approximation method of computing scattering and absorption,” Optics Letters, vol. 22, no. 16, pp. 1205-1207, 1997.

A few useful probability equations

Types of random variables: (pdf, mean, variance, and characteristic function)

Uniform:
$$f_X(x) = \begin{cases} \frac{1}{(b-a)} & a < x < b \\ 0 & \text{otherwise} \end{cases}, \langle X \rangle = \frac{(a+b)}{2}, \sigma_X^2 = \frac{(b-a)^2}{12},$$

$$g(v) = \frac{2}{v(b-a)} \sin\left(\frac{v(b-a)}{2}\right)$$

Gaussian:
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-x_0)^2}{2\sigma^2}}, \langle X \rangle = x_0, \sigma_X^2 = \sigma^2, g(v) = e^{-\left[\frac{(v\sigma)^2}{2}\right]}$$

Exponential:
$$f_X(x) = \begin{cases} \frac{1}{w_0} e^{-\left(\frac{x}{w_0}\right)} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}, \langle X \rangle = w_0, \sigma_X^2 = w_0^2$$

Rayleigh:
$$f_X(x) = \begin{cases} \left(\frac{x}{a}\right) e^{-\left(\frac{x^2}{2a^2}\right)} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}, \langle X \rangle = a\sqrt{\frac{\pi}{2}}, \sigma_X^2 = \frac{\pi}{2}a^2\left(\frac{4}{\pi} - 1\right)$$

Useful integrals: (note $p > 0$ and $n = 0, 1, \dots$)

$$\int_{-\infty}^{\infty} e^{-p^2x^2 \pm qx} dx = \left[e^{\frac{q^2}{4p^2}} \right] \frac{\sqrt{\pi}}{|p|} \quad \int_0^{\infty} x^{2n} e^{-px^2} dx = \frac{(2n-1)(2n-3)\dots(1)}{2(2p)^n} \sqrt{\frac{\pi}{p}}$$

$$\int_0^{\infty} x^{2n+1} e^{-px^2} dx = \frac{n!}{2p^{n+1}} \quad \int_0^{\infty} x^n e^{-px} dx = n! p^{-(n+1)}$$

$$\int_{-\infty}^{\infty} x^n e^{-px^2 + 2qx} dx = \left(\frac{1}{2^{n-1}p}\right) \left(\sqrt{\frac{\pi}{p}}\right) \left[\frac{d^{n-1}}{dq^{n-1}}(qe^{\frac{q^2}{p}})\right]$$