Electromagnetic Kirchhoff Approach for a PEC rough surface

Consider a surface $z = \zeta(x,y)$ excited by an incident field

$$\bar{E}^{inc} = \hat{e}_i e^{i\beta x} e^{-i\gamma z} = \hat{e}_i e^{i\bar{k}_i \cdot \bar{r}} \qquad \bar{k}_i = \hat{x}\beta - \hat{z}\gamma = \hat{x} k\sin\theta_i - \hat{z} k\cos\theta_i$$
$$= \hat{x} k_{ix} - \hat{z} k_{iz}$$

$\hat{e}_i$ here will be chosen from $\hat{h}_i = \hat{y}$ or $\hat{v}_i = -\hat{x}\cos\theta_i - \hat{z}\sin\theta_i$

$$\bar{H}^{inc} = \frac{1}{\eta_0}(\hat{k}_i \times \hat{e}_i) e^{i\bar{k}_i \cdot \bar{r}} \quad \text{with } \hat{k}_i \times \hat{h}_i = -\hat{v}_i, \ \hat{k}_i \times \hat{v}_i = \hat{h}_i$$

The normal to the surface profile is given by $\hat{n} = \dfrac{\hat{z} - \partial\zeta/\partial x\, \hat{x} - \partial\zeta/\partial y\, \hat{y}}{\sqrt{1 + (\partial\zeta/\partial x)^2 + (\partial\zeta/\partial y)^2}}$

Under the Kirchhoff approach, we find the induced currents on the surface using

$$\bar{J}_{po} = 2\hat{n} \times \bar{H}^{inc} = \frac{2}{\eta_0}\left[\hat{n} \times (\hat{k}_i \times \hat{e}_i)\right] e^{ik_{ix}x} e^{-ik_{iz}\zeta}, \quad \bar{M}_{po} = 0$$

Let's consider horizontal incidence, $\hat{e}_i = \hat{h}_i$, then $\hat{k}_i \times \hat{e}_i = -\hat{v}_i$

$$\bar{J}_{po}^{h} = \frac{-2}{\eta_0}(\hat{n} \times \hat{v}_i) e^{ik_{ix}x} e^{-ik_{iz}\zeta}$$
$$= \frac{+2\left(\hat{z} - \partial\zeta/\partial x\,\hat{x} - \partial\zeta/\partial y\,\hat{y}\right) \times (\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)}{\eta_0\sqrt{1 + (\partial\zeta/\partial x)^2 + (\partial\zeta/\partial y)^2}} e^{ik_{ix}x} e^{-ik_{iz}\zeta}$$

$$\bar{J}_{po}^{h} = \frac{2\, e^{ik_{ix}x} e^{-ik_{iz}\zeta}}{\eta_0\sqrt{1 + (\partial\zeta/\partial x)^2 + (\partial\zeta/\partial y)^2}}\left[\hat{x}\left(-\partial\zeta/\partial y \sin\theta_i\right) + \hat{y}\left(\cos\theta_i + \frac{\partial\zeta}{\partial x}\sin\theta_i\right) + \hat{z}\left(\partial\zeta/\partial y \cos\theta_i\right)\right]$$

Next we find the scattered fields in the far field by allowing these currents to radiate in free space... First find vector potential $\bar{A}$ then fields...

$$\bar{A}(\bar{r}) = \frac{\mu_0}{4\pi r} e^{ikr} \iint ds' \, \bar{J}_{ps}(\bar{r}') \, e^{-ik\hat{a}_R \cdot \bar{r}'}$$

here $\hat{a}_R$ is a vector from the origin to a general observation point in the upper hemisphere: $\hat{a}_R = \hat{x}\sin\theta_s\cos\phi_s + \hat{y}\sin\theta_s\sin\phi_s + \hat{z}\cos\theta_s$, while $\bar{r}'$ is a vector from the origin to a point on the surface, $\bar{r}' = \hat{x}x' + \hat{y}y' + \hat{z}\,\zeta(x', y')$. Note also that we can project an integral over the tilted surface ds' onto the x-y plane through the Jacobian of the transformation: $ds' = dx'\,dy'\sqrt{1 + (\partial\zeta/\partial x')^2 + (\partial\zeta/\partial y')^2}$.

Thus, $$\bar{A}(\bar{r}) = \frac{\mu_0}{4\pi r} e^{ikr} \iint_{-\infty}^{\infty} dx'\,dy' \, \frac{2}{\eta_0} e^{ik_{ix}x'} e^{-ik_{iz}\zeta(x',y')} e^{-ik_{sx}x'} e^{-ik_{sy}y'} e^{-ik_{sz}\zeta(x',y')}$$
$$\left[\hat{x}\left(\frac{-\partial\zeta}{\partial y'}\sin\theta_i\right) + \hat{y}\left(\cos\theta_i + \frac{\partial\zeta}{\partial x'}\sin\theta_i\right) + \hat{z}\left(\frac{\partial\zeta}{\partial y'}\cos\theta_i\right)\right]$$

where $k_{sx} = k\hat{a}_R \cdot \hat{x}$, $k_{sy} = k\hat{a}_R \cdot \hat{y}$ & $k_{sz} = k\hat{a}_R \cdot \hat{z}$. We can combine these by defining $k_{dx} = k_{ix} - k_{sx}$, $k_{dy} = -k_{sy}$, $k_{dz} = -k_{iz} - k_{sz}$

to get

$$\bar{A}(\bar{r}) = \frac{2}{\eta_0}\frac{\mu_0}{4\pi r} e^{ikr} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'\,dy' \, e^{ik_{dx}x'} e^{ik_{dy}y'} e^{ik_{dz}\zeta(x',y')} \left[-\hat{x}\frac{\partial\zeta}{\partial y'}\sin\theta_i + \hat{z}\,\partial\zeta/\partial y'\cos\theta_i + \hat{y}\left(\cos\theta_i + \frac{\partial\zeta}{\partial x'}\sin\theta_i\right)\right]$$

Note the term in brackets is also a function of x' & y' due to the $\frac{\partial\zeta}{\partial x'}$ & $\frac{\partial\zeta}{\partial y'}$ variables.

Let's try to get rid of these through an integration by parts:

For example: $$\int_{-\infty}^{\infty} dy' \, e^{ik_{dy}y'} \left[\int_{-\infty}^{\infty} dx' \, e^{ik_{dx}x'} e^{ik_{dz}\zeta(x',y')} \left(+\frac{\partial\zeta}{\partial x'}\sin\theta_i\right)\right]$$

$u = e^{ik_{dx}x'}$, $dv = \frac{\partial\zeta}{\partial x'} e^{ik_{dz}\zeta(x',y')}\,dx'$

$du = ik_{dx}e^{ik_{dx}x'}dx'$, $v = \frac{1}{ik_{dz}} e^{ik_{dz}\zeta(x',y')}$

$$= +\sin\theta_i \int_{-\infty}^{\infty} dy' \left[\frac{e^{ik_{dx}x'} e^{ik_{dz}\zeta(x',y')}}{ik_{dz}}\Bigg|_{-\infty}^{\infty} - \frac{k_{dx}}{k_{dz}}\int_{-\infty}^{\infty} dx' \, e^{ik_{dx}x'} e^{ik_{dz}\zeta(x',y')}\right] e^{ik_{dy}y'}$$

If we neglect the "edge term" by assuming that in reality the currents must taper off somewhere before infinity, we obtain

$$\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy'\, e^{ik_{dx}x'} e^{ik_{dy}y'} e^{ik_{dz}\zeta(x',y')} \left(\frac{\partial \zeta}{\partial x'} \sin\theta_i\right) = \frac{-\sin\theta_i\, k_{dx}}{k_{dz}} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy'\, e^{ik_{dx}x'} e^{ik_{dy}y'} e^{ik_{dz}\zeta(x',y')}$$

Making similar integrations for all the derivative terms we find

$$\bar{A}(\bar{r}) = \frac{2\mu_0}{\eta_0 4\pi r} e^{ikr} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy'\, e^{ik_{dx}x'} e^{ik_{dy}y'} e^{ik_{dz}\zeta(x',y')} \left[\frac{\sin\theta_i\, k_{dy}}{k_{dz}} \hat{x} + \hat{y}\left(\cos\theta_i - \frac{\sin\theta_i\, k_{dx}}{k_{dz}}\right) - \hat{z}\, \frac{\cos\theta_i\, k_{dy}}{k_{dz}} \right]$$

Now recalling that $\bar{E}_s = i\omega \bar{A}_T$ (transverse to $\hat{k}_s$) in the far field, and defining the scattered field polarizations as $\hat{h}_s = -\sin\phi_s \hat{x} + \cos\phi_s \hat{y}$, $\hat{v}_s = \cos\theta_s \cos\phi_s \hat{x} + \cos\theta_s \sin\phi_s \hat{y} - \sin\theta_s \hat{z}$

we find

$$\hat{h}_s \cdot \bar{E}_s = \frac{2ik}{4\pi r} e^{ikr} \left[-\frac{\sin\theta_i\, k_{dy}}{k_{dz}} \sin\phi_s + \cos\phi_s \left(\cos\theta_i - \frac{\sin\theta_i\, k_{dx}}{k_{dz}}\right) \right] I$$

$$\hat{v}_s \cdot \bar{E}_s = \frac{2ik}{4\pi r} e^{ikr} \left[\cos\theta_s \left(\frac{\sin\theta_i\, k_{dy}}{k_{dz}} \cos\phi_s + \sin\phi_s \left(\cos\theta_i - \frac{\sin\theta_i\, k_{dx}}{k_{dz}}\right)\right) + \frac{\cos\theta_i \sin\theta_s\, k_{dy}}{k_{dz}} \right] I$$

with $I = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy'\, e^{ik_{dx}x'} e^{ik_{dy}y'} e^{ik_{dz}\zeta(x',y')}$

Simplifying the terms in [ ] we find:

$$\frac{-\sin\theta_i(+\sin\theta_s \sin\phi_s)\sin\phi_s + \cos\theta_i(\cos\theta_i + \cos\theta_s)\cos\phi_s + \sin\theta_i \cos\phi_s(-\sin\theta_i - \sin\theta_s \cos\phi_s)}{+(\cos\theta_i + \cos\theta_s)}$$

$$= \frac{-\sin\theta_i \sin\theta_s + (1 + \cos\theta_i \cos\theta_s)\cos\phi_s}{(\cos\theta_i + \cos\theta_s)}$$

and $\dfrac{\cos\theta_s(\sin\theta_i)(+\sin\theta_s\sin\phi_s)\cos\phi_s + \sin\phi_s\cos\theta_i(\cos\theta_i+\cos\theta_s)\cos\phi_s + \sin\phi_s\sin\theta_i(\sin\theta_i - \sin\theta_s\cos\phi_s)\cos\phi_s}{+(\cos\theta_i+\cos\theta_s)}$

$$+\frac{\cos\theta_i\sin\theta_s(\sin\theta_s\sin\phi_s)}{\cos\theta_i+\cos\theta_s} = \left(\frac{\sin\phi_s}{\cos\theta_i+\cos\theta_s}\right)\Big(\cos\theta_s + \cos\theta_i\cos^2\theta_s - \sin\theta_i\sin\theta_s\cos\phi_s\cos\phi_s + \sin\theta_i\cos\theta_s\sin\theta_s\cos\phi_s$$
$$+\cos\theta_i\sin^2\theta_s\Big)$$

Thus,

$$\boxed{\begin{aligned} \hat{h}_s\cdot\bar{E}_s &= \frac{2ik}{4\pi r}e^{ikr}\left[\frac{(1+\cos\theta_i\cos\theta_s)\cos\phi_s - \sin\theta_i\sin\theta_s}{\cos\theta_i+\cos\theta_s}\right]I \\ \hat{v}_s\cdot\bar{E}_s &= \frac{2ik}{4\pi r}e^{ikr}\left(\frac{\sin\phi_s}{\cos\theta_i+\cos\theta_s}\right)(\cos\theta_i+\cos\theta_s)I = \frac{2ik}{4\pi r}e^{ikr}(\sin\phi_s)I \end{aligned}}$$

Note the cross $(\hat{v}_s\cdot\hat{E}_s)$ polarized term vanishes for in-plane scattering (including backscatter).

Examining the above, we find for a random surface, the only random quantity is I,

so $\langle\hat{h}_s\cdot\bar{E}_s\rangle$ & $\langle\hat{v}_s\cdot\bar{E}_s\rangle \propto \langle I\rangle$. To find $\langle I\rangle$, write

$$\langle I\rangle = \left\langle \int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dy'\, e^{ik_{dx}x'}e^{ik_{dy}y'}e^{ik_{dz}\zeta(x',y')}\right\rangle$$ & $e^{ik_{dz}\zeta(x',y')}$ is the only

things random $$= \int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dy'\, e^{ik_{dx}x'}e^{ik_{dy}y'}\left\langle e^{ik_{dz}\zeta(x',y')}\right\rangle$$

This is a characteristic function! Denoting $\chi(k_{dz}) = \langle e^{ik_{dz}\zeta(x',y')}\rangle$ we find

$\langle I\rangle$ to be related to the Fourier transform of the surface characteristic function.

To plug in some statistics, let's assume that each point on the surface is a Gaussian

random variable with variance h^2. Note the above is a single point quantity so there

are no stochastic process effects (i.e. correlations) involved. Making this assumption implies that the surface is a <u>Gaussian random process</u> and that all surface statistics can be described in terms of the surface correlation fn (ie. only 2^{nd} moments are required to describe all surface characteristics). We're limiting ourselves by assuming this – all surfaces are not necessarily Gaussian random processes – but it is a reasonable model for many cases.

Now using the characteristic function for a Gaussian random variable:

$$\langle e^{i k_{\Delta z} \zeta(x',y')} \rangle = e^{-(k_{\Delta z} h)^2/2}, \text{ we find}$$

$$\langle I \rangle = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \, e^{i k_{\Delta x} x'} e^{i k_{\Delta y} y'} e^{-(k_{\Delta z} h)^2/2} \quad \leftarrow \text{note Fourier transform of a constant!}$$

$$= e^{-(k_{\Delta z} h)^2/2} (2\pi)^2 \, \delta(k_{\Delta x}) \delta(k_{\Delta y})$$

and $$\langle \hat{h}_s \cdot \bar{E}_s \rangle = \frac{2ik \, e^{ikr}}{4\pi r} e^{-(k_{\Delta z} h)^2/2} \cos\theta_i \, \delta(k_{\Delta x}) \delta(k_{\Delta y}), \quad \langle \hat{v}_s \cdot \bar{E}_s \rangle = 0$$

Note this is the average field which exists only in the specular direction, $k_{sx} = k_{ix}$, $k_{sy} = 0$, and which is reduced according to $e^{-(2kh\cos\theta_i)^2/2}$. Note here we have found a reduced coherent field which gets smaller as the surface roughness increases. The above expression is a little strange because of the δ functions combined with a spherical wave; it can be shown that this expression reduces to a specular plane wave reduced by $e^{-(2kh\cos\theta_i)^2/2}$ if a more careful derivation is performed for a finite size surface that approaches ∞ after the derivation is complete.

Now let's work on the incoherent power: $\langle |\hat{h}_s \cdot \bar{E}_s|^2 \rangle$ & $\langle |\hat{v}_s \cdot \bar{E}_s|^2 \rangle$

again both of which will be proportional to $\langle |I|^2 \rangle$:

$$\langle |\hat{h}_s \cdot \bar{E}_s|^2 \rangle = \frac{4k^2}{16\pi^2 r^2} \left[\frac{(1+\cos\theta_i \cos\theta_s)\cos\phi_s - \sin\theta_i \sin\theta_s}{\cos\theta_i + \cos\theta_s} \right]^2 \langle |I|^2 \rangle$$

$$\langle |\hat{v}_s \cdot \bar{E}_s|^2 \rangle = \frac{4k^2}{16\pi^2 r^2} \sin^2\phi_s \langle |I|^2 \rangle$$

So let's work on $\langle |I|^2 \rangle$:

$$= \left\langle \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dx'' \int_{-\infty}^{\infty} dy'' \, e^{ik_{dx}(x'-x'')} e^{ik_{dy}(y'-y'')} e^{ik_{dz}(\zeta(x',y') - \zeta(x'',y''))} \right\rangle$$

$$= \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dx'' \int_{-\infty}^{\infty} dy'' \, e^{ik_{dx}(x'-x'')} e^{ik_{dy}(y'-y'')} \left\langle e^{ik_{dz}(\zeta(x',y') - \zeta(x'',y''))} \right\rangle$$

which involves a 2 point characteristic function of the random process $\left\langle e^{ik_{dz}(\zeta(x',y') - \zeta(x'',y''))} \right\rangle$

Note this is not a product of 2 single point characteristic fns since our two points may be correlated.

Using our pdf for two gaussian random variables:

$$f_{\zeta'\zeta''}(\zeta',\zeta'') = \frac{1}{2\pi h^2 \sqrt{1-C^2}} \exp\left[-\left(\frac{\zeta'^2 - 2C\zeta'\zeta'' + \zeta''^2}{2h^2(1-C^2)} \right) \right] \quad \text{from PS \#1}$$

recall here C is the correlation coefficient between ζ' & ζ'', we can evaluate the above two point characteristic function (i.e. take $\int d\zeta' d\zeta'' \, f_{\zeta',\zeta''} e^{ik_{dz}(\zeta'-\zeta'')}$)

to find $\left\langle e^{ik_{dz}(\zeta'-\zeta'')} \right\rangle = e^{-(k_{dz} h)^2 (1 - C(\rho))}$

where C has been modified to $C(\rho)$ with $\rho = \sqrt{(x'-x'')^2 + (y'-y'')^2}$ to indicate that the correlation between points on our random surface depends on their spatial separation. Also note we are assuming an isotropic surface since we have $C(\rho)$ not $C(\bar{\rho})$.

Now we have

$$\langle |I|^2 \rangle = \int_{-\infty}^{\infty} dx'' \int_{-\infty}^{\infty} dy'' \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \, e^{i k_{dx}(x'-x'')} e^{i k_{dy}(y'-y'')} e^{-(k_{dz} h)^2 (1 - C(|\bar{r}'-\bar{r}''|))}$$

changing coordinates to $\bar{r}_c = \frac{1}{2}(\bar{r}'+\bar{r}'')$ & $\bar{r}_d = (\bar{r}'-\bar{r}'')$ we obtain

$$= \iint_A d\bar{r}_c \iint_A d\bar{r}_d \, e^{i k_{dx} r_{dx}} e^{i k_{dy} r_{dy}} e^{-(k_{dz} h)^2 (1 - C(|\bar{r}_d|))}$$

where the $-\infty$ to ∞ limits have been replaced by the surface area A to avoid some blowing up problems. Finally, the integral over $\bar{r}_c$ gives A:

$$\langle |I|^2 \rangle = A \iint_A d\bar{r}_d \, e^{i k_{dx} r_{dx}} e^{i k_{dy} r_{dy}} e^{-(k_{dz} h)^2 (1 - C(\bar{r}_d))}$$

Note this includes both incoherent & coherent powers, to remove the coherent power take

$$\langle |I|^2 \rangle - |\langle I \rangle|^2 = A \iint_A d\bar{r}_d \, e^{i k_{dx} r_{dx}} e^{i k_{dy} r_{dy}} \left[e^{-(k_{dz} h)^2 (1 - C(|\bar{r}_d|))} - e^{-(k_{dz} h)^2} \right]$$

whose integrand should approach zero for large $\bar{r}_d$. Modify limits therefore back to ∞ to find

$$\langle |I|^2 \rangle - |\langle I \rangle|^2 = A \iint_{-\infty}^{\infty} d\bar{r}_d \, e^{i k_{dx} r_{dx}} e^{i k_{dy} r_{dy}} \left[e^{-(k_{dz} h)^2 (1 - C(|\bar{r}_d|))} - e^{-(k_{dz} h)^2} \right]$$

We get stuck here until $C(|\bar{r}_d|)$ is specified, at which point the integral may or may not be possible to evaluate analytically.

Now that we've found $\langle |\hat{h}_s \cdot \bar{E}_s|^2 \rangle$ and $\langle |\hat{v}_s \cdot \bar{E}_s|^2 \rangle$ we can determine the bistatic scattering coefficient per unit area as

$$\sigma_{bh}(\hat{k}_s, \hat{k}_i) = \frac{4\pi r^2 \, \frac{1}{2\eta_0} \langle |\hat{b}_s \cdot \bar{E}_s|^2 \rangle}{A \left(\frac{1}{2\eta_0} \right)}$$

Pluging in our expressions for the fields we find

$$\sigma_{hh}(\hat{k}_s,\hat{k}_i) = \underbrace{\frac{4k^2}{16\pi^2 r^2}\,4\pi r^2}_{\frac{k^2}{\pi}} \left[\frac{(1+\cos\theta_i\cos\theta_s)\cos\phi_s - \sin\theta_i\sin\theta_s}{\cos\theta_i+\cos\theta_s}\right]^2 D_I$$

$$\sigma_{vh}(\hat{k}_s,\hat{k}_i) = \frac{k^2}{\pi}\left[\sin^2\phi_s\right] D_I$$

with $D_I = \iint_{-\infty}^{\infty} d\bar{r}_\perp \, e^{i(\bar{k}_d\cdot\bar{r}_\perp)} \, e^{-k_{dz}^2 h^2}\left(e^{k_{dz}^2 h^2 C(|\bar{r}_\perp|)} - 1\right)$

here $\bar{r}_\perp \equiv \hat{x}\, r_{dx} + \hat{y}\, r_{dy}$

At backscattering,

$$\sigma_{hh}(\hat{k}_i) = \sigma_{hh}(-\hat{k}_i,\hat{k}_i) = \frac{k^2}{\pi}\Big/\cos^2\theta_i \, D_I \,, \quad \sigma_{vh} = 0$$

For our gaussian random process, the scattering cross section depends only on the surface variance h^2 & the correlation function $C(|\bar{r}_\perp|)$. For a general $C(|\bar{r}_\perp|)$, the integral in D_I must be evaluated numerically. However if we assume $C(|\bar{r}_\perp|)$ is also Gaussian (a gaussian correlation function) $= e^{-|\bar{r}_\perp|^2/\ell^2}$

then by expanding $e^{k_{dz}^2 h^2 C(|\bar{r}_\perp|)} = 1 + k_{dz}^2 h^2 e^{-|\bar{r}_\perp|^2/\ell^2} + \ldots$

$$= \sum_{m=0}^{\infty} \frac{(k_{dz} h)^{2m} e^{-m|\bar{r}_\perp|^2/\ell^2}}{m!}$$

we find $\iint_{-\infty}^{\infty} d\bar{r}_\perp \, e^{i\bar{k}_d\cdot\bar{r}_\perp} \, e^{-k_{dz}^2 h^2}\left(\sum_{m=1}^{\infty} \frac{(k_{dz} h)^{2m} e^{-m|\bar{r}_\perp|^2/\ell^2}}{m!}\right)$

$$= \pi \sum_{m=1}^{\infty} \frac{(k_{dz}^2 h^2)^m}{m!\, m}\, \ell^2 \exp\left[-\frac{(k_{dx}^2 + k_{dy}^2)\ell^2}{4m}\right] e^{-k_{dz}^2 h^2}$$

and we have a sum instead of an integral!

Going back to the integral for arbitrary $C(|\bar{r}_\perp|)$, we can see that for $k_{dz}h \ll 1$, we can approximate

$$D_I \approx \iint_{-\infty}^{\infty} d\bar{r}_\perp \, e^{i(\bar{k}_d \cdot \bar{r}_\perp)} (1 - k_{dz}^2 h^2)\left(k_{dz}^2 h^2 \, C(\bar{r}_\perp) \right)$$

$$= k_{dz}^2 h^2 \iint_{-\infty}^{\infty} d\bar{r}_\perp \, e^{i(\bar{k}_d \cdot \bar{r}_\perp)} C(\bar{r}_\perp) \leftarrow \text{Fourier transform of correlation fn, eval at } \bar{k}_d$$

$\propto k_{dz}^2 \, W(k_{xi} - k_{xs}, k_{yi} - k_{ys})$ as in the SPM! However, the angular dependent terms out front differ for some polarization quantities. This discrepancy between PO & SPM in the small height limit was considered a "controversy" by some, but since has been resolved to show that the SPM yields the correct result.

We can also approximate D_I in the limit that $k_{dz}h \gg 1$. In this case,

$$D_I = \iint_{-\infty}^{\infty} d\bar{r}_\perp \, e^{i(\bar{k}_d \cdot \bar{r}_\perp)} e^{-k_{dz}^2 h^2}\left(e^{k_{dz}^2 h^2 C(|\bar{r}_\perp|)} - 1 \right)$$

will fall off extremely rapidly for $|\bar{r}_\perp| > 0$, so it is reasonable to replace $C(|\bar{r}_\perp|)$ with

$$\left(1 - \left| \frac{d^2}{d|\bar{r}_\perp|^2} C(\bar{r}_\perp) \right|_{|\bar{r}_\perp|=0} \frac{|\bar{r}_\perp|^2}{2} \right),$$

i.e. a power series to 2nd order in $|\bar{r}_\perp|$. Note we don't need a first order term here because $C(|\bar{r}_\perp|) \propto \langle z(\bar{r}_1) z(\bar{r}_1 + \bar{r}_\perp) \rangle$ should be symmetric at $\bar{r}_\perp = 0$ & therefore cannot have a linear term.

Using this expansion & rewriting $d\bar{r}_\perp = \rho \, d\rho \, d\phi$, we find

$$D_I = \int_0^\infty d\rho \int_0^{2\pi} d\phi \, \rho \, e^{i(\bar{k}_d \cdot \bar{r}_\perp)} e^{-k_{dz}^2 h^2} \left(e^{k_{dz}^2 h^2} e^{-k_{dz}^2 h^2 |C''(0)| \rho^2 / 2} - 1 \right)$$

$$D_I = \int_0^\infty d\rho\, \rho \left(e^{-k_{dz}^2 h^2 |C''(0)| \rho^2/2} - e^{-k_{dz}^2 h^2} \right) \int_0^{2\pi} d\phi\, e^{i k_{d\rho} \rho \cos(\phi - \phi_k)}$$

where $\bar{k}_d = \hat{x} k_{d\rho} \cos\phi_k + \hat{y} k_{d\rho} \sin\phi_k$, i.e. $k_{d\rho} = \sqrt{k_{dx}^2 + k_{dy}^2}$, $\phi_k = \tan^{-1}\left(\frac{k_{dy}}{k_{dx}}\right)$

$$= \int_0^\infty d\rho\, \rho \left(e^{-k_{dz}^2 h^2 |C''(0)| \rho^2/2} \underbrace{- e^{-k_{dz}^2 h^2}}_{\text{neglect}} \right) \left(2\pi J_0(k_{d\rho} \rho) \right)$$

From an integral table, we can find

$$= 2\pi \frac{1}{k_{dz}^2 h^2 |C''(0)|} e^{-\frac{k_{d\rho}^2}{2 k_{dz}^2 h^2 |C''(0)|}}$$

$$= \frac{2\pi}{k_{dz}^2 h^2 |C''(0)|} e^{-\frac{k_{d\rho}^2}{2 k_{dz}^2 h^2 |C''(0)|}}$$

Thus, in the limit of very large roughness we obtain no coherent field ($e^{-k_{dz}^2 h^2} \simeq 0$) and the above expression for D_I in terms of $|C''(0)| = \left| \frac{\partial^2 C(|\bar{r}_\perp|)}{\partial |\bar{r}_\perp|^2} \Big|_{|\bar{r}_\perp| = 0} \right|$

It turns out that it can be shown that $h^2 |C''(0)|$ is the variance of the surface slopes, s^2. that is the average of the slope squared over the surface, and D_I

$$\text{is} = \frac{2\pi}{k_{dz}^2 s^2} e^{-\frac{k_{d\rho}^2}{2 k_{dz}^2 s^2}}$$

This result is called the "geometrical optics" limit (since it applies as $k \to \infty$) and can be interpreted as relating the power scattered at a specific angle to the probability of obtaining a point on the surface oriented in the correct direction to produce a specular reflection.