QUIZ 1
ECE 816: Spring quarter 2009

Problem 1: A plane wave propagating in the $\hat{z}$ direction with an electric field $\overline{E}_i = [(2-i)\hat{x} + \hat{y}]e^{ikz}$ is incident upon a small dielectric sphere of the radius $a = 0.5$ mm and relative permitivity $\epsilon_r = 1.5$. This wave is generated by a source whose frequency satisfies the condition $ka \ll 1$.

a) The extinction cross section of this particle is 2.6×10^{-15} m^2. What is the frequency of the incident wave?

Since the dielectric sphere is lossless ($\mathcal{I}m\{\epsilon_r\} = 0$), the absorption cross section $\sigma_a = 0$. Thus the extinction cross section $\sigma_t = \sigma_s$, the scattering cross section.

$$\sigma_t = \sigma_s = \frac{128\pi^5 a^6}{3\lambda^4}\left|\frac{\epsilon_r - 1}{\epsilon_r + 2}\right|^2$$

Solving for λ gives $\lambda = 20$ cm; frequency $f = c/\lambda =$**1.5 GHz**.

b) Find the modified Stokes vector of the backscattered field. Specify your polarization definitions; they should be chosen so that $\hat{a}_1 \times \hat{a}_2 = \hat{k} = -\hat{z}$. Your answer should be a function of the range R at which the scattered field is observed.

The backscattered field is

$$\overline{E}_s(z = -R) = \overline{f}(\hat{O} = -\hat{z}, \hat{i} = \hat{z}) = [(2 - i)\hat{x} + \hat{y}]|\overline{f}|\frac{e^{ikR}}{R}$$

where

$$|\overline{f}| = \frac{k^2}{4\pi}\frac{3(\epsilon_r - 1)}{\epsilon_r + 2}(\frac{4}{3}\pi a^3) \simeq 1.76 \times 10^{-8}$$

Choose $\hat{a}_1 = \hat{x}$ and $\hat{a}_2 = -\hat{y}$ such that $\hat{a}_1 \times \hat{a}_2 = -\hat{z}$:

$$\overline{E}_s(z = -R) = (a_1\hat{a}_1 + a_2\hat{a}_2)e^{-ikz}$$

Hence, $a_1 = \frac{2-i}{R}|\overline{f}|$, $a_2 = -\frac{1}{R}|\overline{f}|$, and

$$\begin{aligned}
\mathbf{I_1} &= |a_1|^2 = \mathbf{\frac{5}{R^2}|\overline{f}|^2} \\
\mathbf{I_2} &= |a_2|^2 = \mathbf{\frac{1}{R^2}|\overline{f}|^2} \\
\mathbf{U} &= 2\,\mathcal{R}e\{a_1 a_2^*\} = -\mathbf{\frac{4}{R^2}|\overline{f}|^2} \\
\mathbf{V} &= 2\,\mathcal{I}m\{a_1 a_2^*\} = \mathbf{\frac{2}{R^2}|\overline{f}|^2}
\end{aligned}$$

Problem 2: Consider two independent (and uncorrelated) Gaussian random variables, X and Y, each with zero mean and variance σ^2. Find the probability density function of $X^2 + Y^2$.

From PS #1, problem 1 solution, the quantity $R = \sqrt{X^2 + Y^2}$ is a Rayleigh distributed random variable with the pdf

$$f_R(r) = \frac{r}{\sigma^2}e^{-r^2/(2\sigma^2)}$$

To find the pdf of the power $P = R^2$, we can follow an example in lecture 2 (pg. 13):

$$f_P(p) = \frac{d}{dp}\left[\int_0^{\sqrt{p}} dr\, f_R(r)\right] = \frac{d}{dp}\left[1 - e^{-p/(2\sigma^2)}\right] = \mathbf{\frac{1}{2\sigma^2}e^{-p/(2\sigma^2)}}$$