EE 816 - Lecture 1

1. Administrative
2. Deterministic versus “random” media
3. Fields as random variables
4. An example problem
5. Overview of course

II. Deterministic versus “random” media

- By a deterministic problem, we mean the configuration and properties of scatterers are completely known
- We are familiar with electromagnetic solutions for deterministic scattering problems
- Methods are known but obtaining a solution can be tough!
- For a “random medium” there is an additional complexity: configuration and/or properties of scatterers are not known exactly but only in terms of their statistics

Why describe media as random?

- In many realistic problems, there are too many scatterers to specify all properties completely
- Scatterer statistics may not be that complicated however
- In some cases, we can obtain a good understanding of fields in the presence of these scatterers just from the random medium approach
- A few examples:
  - changing dielectric properties of the atmosphere which can affect wave propagation
  - reflection and scattering from the surface of the ocean
  - scattering and propagation through foliage

  All of these media have numerous scatterers which would be difficult to describe exactly!

III. Fields as random variables

- Note however that there is a fundamental difference between deterministic and random scattering problems: finding exact fields versus finding field statistical properties
- Since the scatterers themselves are not precisely known, we do not obtain precise scattered fields, but rather descriptions of their average behavior
- We can still apply a standard statistical description to scattered fields however: completely characterized by knowledge of the probability density function
- In many cases this would be diffcult to find and we will be satisfied with moments: average, variance, etc.
- Very fundamental: in a random medium problem we do not obtain “exact field” solutions but rather a description of field statistics

- Since scattered fields are complex quantities, we will be dealing primarily with complex random variables
- We can also talk about statistics of radar cross sections, or more commonly radar cross sections per unit area or volume
- Just like radar cross sections, scattered field statistics can be functions of incidence and scattering angles, frequency, polarization, scatterer properties, etc.
- The complexity of typical scattering problems is going to limit our studies: exact solutions will usually be too complicated to do interesting statistical studies
- We'll wind up studying field statistics obtained from approximate scattering solutions: PO, Rayleigh scattering, etc.
- Be sure to remember what approximations are being made!

IV. An example problem

- Consider a scattering problem in which a plane wave impinges on a perfectly conducting sphere
- However, the problem is random because the sphere is only present 10% of the time, otherwise the plane wave is not scattered
- Clearly in this case we will only get a backscattered field 10% of the time, otherwise we will measure zero
- Thus, the average backscattered field measured over all measurements would be 10% of the sphere backscattered field
- Although we know the average field, on a given try this will not be a great guess because really we either get 100% of the backscattered field or 0%
- We could also compute higher order moments of the field to get a better idea of the nature of this problem

V. Overview of course

- We will study several approximate methods for predicting field statistics, both in the random medium and rough surface problems
- 1st two weeks: review scattering and probability theory
- 3rd week: independent scattering theory (discrete scatterers)
- 4th-6th week: radiative transfer theory (discrete scatterers)
- 7th week: continuous random medium theory (continuous scatterers)
- 8th-10th week: surface scattering theory (rough surface)
- Computer project: a numerical test of one of these scattering theories

EE 816 - Lecture 2

1. Review probability theory
2. Random variables
3. Moments of random variables
4. Collections of random variables
5. "Monte Carlo" techniques

I. Review probability theory

- Probability theory begins fundamentally with a “probability space”, comprised of a “sample space” containing all possible outcomes of an experiment, an “event space” containing all subsets of the sample space to which a probability will be assigned, and a “probability measure” which assigns a number between 0 and 1 to every set in the event space
- Essentially we are describing an experiment for which the outcome is fundamentally unknown, but we can describe the “likelihood” of a certain outcome if we perform the experiment a great number of times
- The probability measure must have certain properties: the number P assigned is always $0 \leq P \leq 1$, and the probability of obtaining an outcome is 1.

- Another property: if events F and G do not intersect (i.e. do not have any outcomes in common) then $P(F \cup G) = P(F) + P(G)$, where $\cup$ indicates the unions of sets F and G.
- If F and G are not “disjoint” then $P(F \cup G) = P(F) + P(G) - P(F \cap G)$, where $\cap$ indicates “intersection”, i.e. the common elements of F and G
- There is a lot of abstract mathematics underlying probability theory, but we will go with an intuitive approach; has its limits but will usually be ok in our problems

II. Random variables

- A random variable is an unknown quantity which can be described statistically
- A random variable can be the outcome of an experiment, or a function of the outcome of an experiment
- Can be continuous (taking on a continuous value) or discrete (taking only a finite set of values)
- Examples:
  - flip of a coin (discrete)
  - uniformly distributed real number from 0 to 1 (continuous)
  - noise voltage output of a resistor (continuous)

- All information about a single random variable is contained in the probability density function (or pdf); called probability mass function (pmf) if discrete
- The probability of obtaining a value of the continuous random variable X between values a and b is given by

$$P(a < X \leq b) = \int_a^b f_X(x)dx \tag{1}$$

where f_X is the pdf of random variable X; defines pdf

- The distribution function of a random variable is the integral of f_X:

$$F_X(a) = \int_{-\infty}^a f_X(x)dx \tag{2}$$

and describes the probability that $X \leq a$.

- Common pdf’s: uniform, gaussian, exponential
- We must have $\int_{-\infty}^{\infty} f_X(x)dx = 1$, and f_X cannot be negative

A typical problem: find pdf of a function of X with known f_X.

Example: X is a uniformly distributed real number between 0 and 1. Find pdf for $Y = X^2$.

$$F_Y(y_0) = P(Y \le y_0) = P(X^2 \le y_0) \tag{3}$$

$$P(X \le \sqrt{y_0}) = \int_0^{\sqrt{y_0}} dx = \sqrt{y_0} \tag{4}$$

$$f_Y(y_0) = \frac{d\sqrt{y_0}}{dy_0} = \frac{1}{2\sqrt{y_0}} \tag{5}$$

- Thus the pdf of Y is $\frac{1}{2\sqrt{y}}$ for $0 < Y < 1$
- This is a general procedure: first for $Y = G(X)$, express $F_Y(y_0) = P(Y \le y_0) = P(G(X) \le y_0)$
- Then find $P(X \le G^{-1}(y_0))$ which determines F_Y, derivative is f_Y
- Working it out, this gives $f_Y(y_0) = f_X(G^{-1}(y_0)) \left|\frac{d}{dy_0} G^{-1}(y_0)\right|$ if these functions are well behaved

III. Moments of random variables

- Although the pdf provides a complete description of a single RV, in many cases we will not work with this much information
- Simpler characterizations are in terms of moments: average, variance, etc.
- The "expected value" (or average) of a random variable or function of a random variable is given by

$$< h(X) > = \int_{-\infty}^{\infty} h(x) f_X(x) dx \tag{6}$$

- For example, the expected value of RV X uniformly distributed between 0 and 1 is

$$< X > = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 x dx = 1/2 \tag{7}$$

as expected

- The second "central" moment of a random variable is called the variance and given by

$$< (X - < X >)^2 > = \int_{-\infty}^{\infty} (X - < X >)^2 f_X(x) dx \tag{8}$$

- The "standard deviation" of a random variable is defined to be the square root of the variance
- The variance (and standard deviation) is a measure of the expected "deviation" of a random varaible from its mean
- Higher order moments are defined similarly
- Note if the pdf of a RV is known, all of its moments can be determined; however knowledge of a finite number of moments does not provide pdf

- Notice also if we are talking about a complex valued random variable X we can take different moments: $< X >$, $< |X| >$, $< |X|^2 >$, $| < X > |^2$
- An interesting function is the "characteristic function", given by

$$g(\nu) = < e^{i\nu(X - <X>)} > = \int_{-\infty}^{\infty} e^{i\nu(x - <X>)} f_X(x) dx \tag{9}$$

which is related to the Fourier transform of the pdf

- The characteristic function is interesting because, if expanded in a power series about $\nu = 0$, individual terms are proportional to the moments of X

IV. Collections of random variables

- In many problems there will be more than one quantity which is random; need multiple random variables
- The complete description of these type problems involves the joint probability density function of all the variables together
- For example: in a problem with 2 RV's X and Y, we need to know $f_{X,Y}(x,y)$ to completely describe things
- We can also talk about joint moments; the moment $<(X-<X>)(Y-<Y>)>$ is called a covariance between two RV's
- RV's for which the covariance is zero are called "uncorrelated"
- RV's are called "independent" if $f_{X,Y}(x,y) = f_X(x) f_Y(y)$
- RV's can be uncorrelated but not independent!

- As we add more and more RV's we need a higher and higher dimensional joint pdf; this can get complicated
- We can still define joint moments, calculate expected values, a characteristic function, etc.
- The limiting case of an infinite number of random variables is a "stochastic process"
- For example, consider a time varying function $x(t)$ which is unknown
- The value of $x(t)$ at a particular time t_0 is a random variable
- When we consider all times, we have an infinite number of random variables, or a stochastic process
- There exist differing averages: "ensemble" versus "time" average
- A stochastic process can also be a function of space, like a random surface profile $z(x)$

- Typically we will describe stochastic processes only in terms of their first and second moments
- By this we mean the moments at a single point $<x(t_0)>$ or $<x(t_0)^2>$ and also the joint moments between points $<x(t_0)x(t_1)>$
- The second quantity is a "covariance function" $\Gamma(t_0, t_1)$
- If the random process is "stationary", meaning that its statistical properties do not depend on time, then $\Gamma(t_0, t_1)$ depends only on the time difference $t_1 - t_0$
- Defining $\tau = t_1 - t_0$, we have $\Gamma(\tau) = <x(t_0)x(t_0+\tau)>$
- It can be shown that the "power spectral density" of the random process is given by the Fourier transform of $\Gamma(\tau)$: this is a measure of the frequency components of the process
- Note again this is only a second order moment characterization, a limited description of a stochastic process

V. "Monte Carlo" techniques

- We have learned how to write the expected value of a function of RV X as $<g(X)>$ or in terms of an integral over the pdf
- Ideally we would like to evaluate this integral analytically; if not possible we can evaluate numerically perhaps
- A "Monte Carlo" approach is an alternative technique for evaluating moments of random variables
- Generate many samples of RV X, called X_n, evaluate $g(X_n)$ for each sample, the take average of $g(X_n)$'s to estimate $<g(X)>$
- Should reduce to correct $<g(X)>$ if n approaches ∞
- We can test this by comparing values of $<g(X)>$ obtained from larger and larger values of n to see if we are getting convergence
- Especially useful for complicated problems where it may be hard to even write down an integral!

EE 816 - Lecture 3

1. Definitions of RCS and scattering amplitudes
2. General properties of cross sections
3. Forward scattering theorem
4. Integral representations

I. Definitions of RCS

- Let's review the single particle scattering problem, we'll need to understand this one to talk about many particles
- Consider a plane wave in free space which illuminates a scatterer:

$$\overline{E}_i(\overline{r}) \quad = \quad \hat{e}_i e^{ik\hat{\imath}\cdot\overline{r}} \tag{10}$$

- This is a field polarized in the $\hat{e}_i$ direction and propagating with wavenumber $k = \omega\sqrt{\mu_0\epsilon_0} = 2\pi/\lambda$ in direction $\hat{\imath}$
- Note Ishimaru uses an $e^{-i\omega t}$ time convention: change j's to $-i$'s. We'll adopt his notation too to avoid confusion.
- The scatterer is an object with relative dielectric constant $\epsilon_r(\overline{r}) = \epsilon_r'(\overline{r}) + i\epsilon_r''(\overline{r})$

- In the far field of the scatterer, the scattered field can be written as

$$\overline{E}_s(\overline{r}) \quad = \quad \overline{f}(\hat{O},\hat{\imath})\frac{e^{ikR}}{R} \tag{11}$$

which is an outgoing spherical wave with an amplitude and field direction determined by $\overline{f}$ which depends on both incidence ($\hat{\imath}$) and scattering ($\hat{O}$) propagation directions

- $\overline{f}(\hat{O},\hat{\imath})$ is called the "scattering amplitude"
- Note even though the incident field is linearly polarized, in general the scattered field is elliptically polarized
- The bistatic radar cross section is defined as:

$$\sigma_{bi}(\hat{O},\hat{\imath}) \quad = \quad \lim_{R\to\infty}\frac{4\pi R^2 S_s}{S_i} = 4\pi\left|\overline{f}(\hat{O},\hat{\imath})\right|^2 \tag{12}$$

where S_s and S_i are the time average Poynting powers of the scattered and incident fields respectively - has units of area

- The backscattered RCS is $\sigma_{bi}(-\hat{\imath},\hat{\imath})$

- We can also consider the total amount of power scattered by integrating $\frac{\sigma_{bi}}{4\pi}$ over all scattering angles:

$$\sigma_s(\hat{\imath}) \quad = \quad \frac{1}{4\pi}\int d\theta_0 \int d\phi_0 \sin\theta_O \; \sigma_{bi}(\hat{O},\hat{\imath}) \tag{13}$$

which is known as the scattering cross section

- Note if the scatterer is composed of a lossy dielectric we can lose power inside the dielectric. Define an "absorption cross section" through

$$\sigma_a(\hat{\imath}) \quad = \quad \frac{P_a}{S_i} \tag{14}$$

which also has units of area

- The extinction (or total) cross section includes both scattering and absorption effects:

$$\sigma_t(\hat{\imath}) \quad = \quad \sigma_a(\hat{\imath}) + \sigma_s(\hat{\imath}) \tag{15}$$

- A few more definitions:

$$\sigma_d = \frac{\sigma_{bi}(\hat{O},\hat{i})}{4\pi} \quad (16)$$

$$p(\hat{O},\hat{i}) = \sigma_{bi}(\hat{O},\hat{i})/\sigma_t \quad (17)$$

$$W_0 = \frac{\sigma_s}{\sigma_t} \quad (18)$$

  where σ_d is the differential scattering cross section (area), $p(\hat{O},\hat{i})$ is known as the "phase function" (unitless), and W_0 is known as the "scattering albedo" (unitless)

- Note by combining the absorption and scattering cross sections into the extinction cross section, we should be able to find some kind of conservation of power result
- All power modified by presence of scatterer must be either scattered or absorbed

II. General Properties of Cross Sections

- For a particle much larger than the EM wavelength, the total cross section approaches to twice its geometrical cross sectional area (σ_g) along the line of sight
- This is due to the shadowed region behind the particle: scattered fields cancel incident field there, and incident power redirected elsewhere
- For large lossy scatterers, the absorption cross section can approach but not exceed σ_g
- It can be shown for scatterers much smaller than the EM wavelength that $\sigma_s \propto V^2/\lambda^4$, where V is the volume of the scatterer - we will study in our Rayleigh scattering section
- It can also be shown for a large scatterer that the backscattering cross section reduces to $\pi a_1 a_2 |\Gamma|^2$, where a_1 and a_2 are the radii of curvature at the specular point

III. Forward Scattering Theorem

- The forward scattering theorem provides a relationship between the extinction (or total) cross section σ_t and the scattering amplitude in the forward direction $\overline{f}(\hat{i},\hat{i})$:

$$\sigma_t = \frac{4\pi}{k}\text{Im}\left\{\overline{f}(\hat{i},\hat{i})\cdot\hat{e}_i\right\} \quad (19)$$

- It is clear there should be some relationship between these quantities, since with no scatterer present, all incident power continues to propagate in the forward direction
- This can be a very useful theorem: from just knowing forward scattered fields we can find the extinction cross section
- However we have to be careful if we only know an approximate forward scattered field
- See Born and Wolf or Tsang, Kong, and Shin for proof

IV. Integral representations

- In general, determining the RCS of a scatterer can be difficult, and usually we will make some kind of approximation
- However, we can write down formal expressions for scattering and absorption cross sections in terms of the internal fields in a particle - if known exactly, we get exact cross sections
- These ideas are based on the volume equivalence principle: replace a dielectric scatterer with permittivity $\epsilon_r(\overline{r})$ with an equivalent volume current source

$$\overline{J}_{eq} = -i\omega\epsilon_0\left[\epsilon_r(\overline{r})-1\right]\overline{E}(\overline{r}) \quad (20)$$

  where $\overline{E}$ is the field inside the scatterer. Allowing this current to radiate in free space produces scattered fields.

- It can be shown that:

$$\overline{f}(\hat{O},\hat{i}) = \frac{k^2}{4\pi}\iiint_V \left\{-\hat{O}\times\hat{O}\times\overline{E}(\overline{r}')\right\}\{\epsilon_r(\overline{r}')-1\}\exp\{-ik\hat{O}\cdot\overline{r}'\}dV'$$

- We can find the absorption cross section by integrating $\epsilon''\left|\overline{E}\right|^2$ over the scatterer volume to determine the power loss
- It can be shown that

$$\sigma_a = \iiint_V k\epsilon_r''(\overline{r}')\left|\overline{E}(\overline{r}')\right|^2 dV' \qquad (21)$$

- Again these equations require us to know the electric field inside the particle