EE 816 - Lecture 4

1. Rayleigh scattering
2. Dipole interpretation
3. Cross sections
4. Other approximations

I. Rayleigh Scattering

- Rayleigh scattering is an approximation used to predict scattering from particles much smaller than an electromagnetic wavelength
- The approximation is based on quasi-static ideas: in regions of space much smaller than a wavelength, fields act approximately like static fields
- Basic procedure: solve statics problem to determine field inside scatterer when a uniform field (plane wave) impinges from outside
- This determines field inside particle so we can then compute scattering cross sections, etc, using our previous equations
- This is a nice method because we have lots of analytical techniques for solving the Laplace equation, also usually will predict constant fields inside scatterer

- Let's try it for a sphere: from statics we find that a uniform field $\overline{E}_i$ applied to a dielectric sphere produces an internal field given by:

$$\overline{E} = \frac{3}{\epsilon_r + 2}\overline{E}_i \qquad (1)$$

- Note since the volume of the sphere is small compared to lambda, we can neglect any phase variations $\overline{E}_i$ and approximate the internal field as a constant
- Now use our equations

$$\overline{f}(\hat{O},\hat{i}) = \frac{k^2}{4\pi}\iiint_V \left\{-\hat{O}\times\hat{O}\times\overline{E}(\overline{r}')\right\}\{\epsilon_r(\overline{r}')-1\}\exp\{-ik\hat{O}\cdot\overline{r}'\}dV'$$

- We find

$$\overline{f}(\hat{O},\hat{i}) = \frac{k^2}{4\pi}\frac{3(\epsilon_r-1)}{\epsilon_r+2}V\left[-\hat{O}\times\hat{O}\times\hat{e}_i\right] \qquad (2)$$

where V is the volume of the sphere

II. Dipole interpretation

- Using the $\overline{a}\times\overline{b}\times\overline{c} = \overline{b}(\overline{a}\cdot\overline{c}) - \overline{c}(\overline{a}\cdot\overline{b})$ rule, we can simplify to

$$\left|\overline{f}(\hat{O},\hat{i})\right| = \frac{k^2}{4\pi}\frac{3(\epsilon_r-1)}{\epsilon_r+2}V\sin\chi \qquad (3)$$

  where χ is the angle between $\hat{e}_i$ and $\hat{O}$
- If we remember that these relationships came from the volume equivalence theorem, it is seen that these fields are radiated by an equivalent current $\overline{J}_{eq} = -i\omega\epsilon_0(\epsilon_r - 1)\overline{E}$ which is in the same direction as the incident field
- This is a small current source, effectively a point dipole in the same direction as the incident field
- Thus the scattered field is that of a point dipole oriented in the direction of the incident field with an amplitude which is a function of ϵ_r and V; also varies as k^2

- We can get a few interesting insights by realizing that small particles scatter like dipoles pointing in the direction of the incident field and have a k^2 amplitude
- k^2 means that we obtain significantly larger scattered fields at higher frequencies: used to explain why the sky appears blue
- Also can explain why solar radiation observed straight up appears polarized: due to orientation of dipoles
- Rayleigh scattering can also be derived for non-spherical shapes, similar properties but different field amplitudes inside, also may depend on polarization
- Remember Rayleigh scattering only applies for particles much smaller than the EM wavelength: Ishimaru quotes a radius of 0.05λ as the limit

III. Cross sections

- From knowing the scattering amplitudes we can determine the bistatic, scattering, and also absorption cross sections from the internal fields
- Results are:

$$\sigma_{bi}(\hat{O},\hat{i}) = \frac{k^4}{4\pi}\left|\frac{3(\epsilon_r-1)}{\epsilon_r+2}\right|^2 V^2 \sin^2\chi \tag{4}$$

$$\sigma_s = \frac{128\pi^5 a^6}{3\lambda^4}\left|\frac{\epsilon_r-1}{\epsilon_r+2}\right|^2 = \frac{3k^4V^2}{2\pi}\left|\frac{\epsilon_r-1}{\epsilon_r+2}\right|^2 \tag{5}$$

$$\sigma_a = k\epsilon_r''\left|\frac{3}{\epsilon_r+2}\right|^2 V \tag{6}$$

- Note the RCS varies as k^4a^6 which is a general property of Rayleigh scattering - increasing as f^4. Note units of m^2 also
- σ_s was determined by integrating σ_{bi} over all scattered angles

- σ_a was determined by integrating power lost over volume of scatterer
- Plots comparing σ_s and σ_a to geometrical cross section πa^2 show increase in frequency
- σ_t found from $\sigma_a + \sigma_s$. Note we could also use the optical theorem, but since we are using an approximation it turns out the answer for σ_t is not very accurate
- Note we are talking about f being in the same direction as $\overline{E}_i$ here, but in the far field we should really resolve f into $\hat{\theta}$ and $\hat{\phi}$ components
- For an incident field polarized in the $\hat{x}$ direction,

$$E_\theta = E_0\left(\cos\theta\cos\phi\right)\exp\left(ikR\right) \tag{7}$$

$$E_\phi = E_0\left(-\sin\phi\right)\exp\left(ikR\right) \tag{8}$$

where

$$E_0 = \frac{k^2}{4\pi}\left|\frac{3(\epsilon_r-1)}{\epsilon_r+2}\right|^2\frac{V}{R} \tag{9}$$

- We can now look at Rayleigh scattering in a more general way: an applied field produces a dipole moment inside the scatterer which re-radiates dipole fields
- Thus the fundamental relationship describing the scatterer is the “polarizability” α, defined so that

$$\overline{P} = \alpha\overline{E}_i \tag{10}$$

where $\overline{P}$ is the induced dipole moment (dimensions C-m)

- Scattered fields from $\overline{P}$ are

$$\overline{E} = -\frac{k^2 e^{ikR}}{4\pi\epsilon_0 R}\left|\overline{P}\right|\left\{\hat{O}\times\hat{O}\times\hat{P}\right\} \tag{11}$$

- Scattering cross sections, etc., can therefore be computed in terms of α
- For non-spherical particles, we need a polarizability matrix
- Basis of “discrete dipole approximation” (DDA) computer project

IV. Other approximations

- Ishimaru also describes some other approximate techniques for determining scattering from a single particle
- Born approximation: applies to moderate size particles as long as the contrast to the outer medium is small, i.e. $\epsilon_r \approx 1$. Approximate field inside scatterer is incident field.
- WKB approximation: field inside scatterer approximated as transmission coefficient propagating at wave velocity inside scatterer. Applies to small contrast but larger scatterers
- Ishimaru also discusses Mie theory, which is another name for the exact eigenfunction solution for scattering from a sphere. Sometimes used with larger size particles in random medium theory

EE 816 - Lecture 5

1. Polarization effects
2. Stokes parameters
3. Natural light
4. Polarized scattering

I. Polarization effects

- So far we've just been considering an unspecified linear polarization as the incident field in our scattering problems
- Since scattering can depend on polarization for non-spherical objects, we need to be more specific in general
- The most specific RCS is in terms of both incident and scattered field polarizations: HH, HV, VH, VV
- Note in our σ_s, σ_t, and σ_a definitions, we added powers in all scattered polarizations; these depend only on incident polarization
- We will need a way of describing a general elliptically polarized plane wave
- We can do this with usual field quantities, but it is more common in random medium theory to discuss the "Stokes parameters"

II. Stokes parameters

Consider a plane wave propagating in the z direction:

$$\begin{aligned} E_x &= \mathrm{Re}\left\{E_1 e^{-i\omega t}\right\} = \mathrm{Re}\left\{a_1 \exp(-i\delta_1 + ikz - i\omega t)\right\} = a_1 \cos(\tau + \delta_1) \\ E_y &= \mathrm{Re}\left\{E_2 e^{-i\omega t}\right\} = \mathrm{Re}\left\{a_2 \exp(-i\delta_2 + ikz - i\omega t)\right\} = a_2 \cos(\tau + \delta_2) \\ E_z &= 0 \end{aligned} \tag{12}$$

where $\tau = \omega t - kz$. If we plot the field vector in the time domain at a fixed point in space we will obtain an ellipse

$$\left[E_x/a_1\right]^2 + \left[E_y/a_2\right]^2 - (2E_xE_y/a_1a_1)\cos(\delta) = \sin^2\delta \tag{13}$$

where $\delta = \delta_2 - \delta_1$. It can be shown that $\sin\delta > 0$ is left handed and $\sin\delta < 0$ is right handed.

Note here that Ishimaru defines his phasors as $a_1 \exp(-i\delta_1)$; also for general coordinate systems choose $\hat{a}_1 \times \hat{a}_2 = \hat{i}$ where $\hat{i}$ is the direction of propagation

The above field description is adequate, but another notation was introduced by G. G. Stokes:

$$\begin{aligned} I &= a_1^2 + a_2^2 = |E_1|^2 + |E_2|^2 && (14)\\ Q &= a_1^2 - a_2^2 = |E_1|^2 - |E_2|^2 && (15)\\ U &= 2a_1 a_2 \cos\delta = 2\mathrm{Re}\left\{E_1 E_2 *\right\} && (16)\\ V &= 2a_1 a_2 \sin\delta = 2\mathrm{Im}\left\{E_1 E_2 *\right\} && (17) \end{aligned}$$

from which it can be shown that

$$I^2 = Q^2 + U^2 + V^2 \qquad (18)$$

so that a general elliptically polarized field can be expressed in terms of three parameters only.

Example: linear, amp E_0 at angle Ψ_0 with respect to x axis

$$I = E_0^2,\ Q = E_0^2 \cos 2\Psi_0,\ U = E_0^2 \sin 2\Psi_0,\ V = 0 \qquad (19)$$

right hand circular polarized, amp E_0

$$I = 2E_0^2,\ Q = 0,\ U = 0,\ V = -2E_0^2 \qquad (20)$$

There also is another set of "modified Stokes parameters"

$$\begin{aligned} I_1 &= |E_1|^2 && (21)\\ Q_1 &= |E_2|^2 && (22)\\ U &= 2\mathrm{Re}\left\{E_1 E_2 *\right\} && (23)\\ V &= 2\mathrm{Im}\left\{E_1 E_2 *\right\} && (24) \end{aligned}$$

Also note that it can be easier to plot the polarization ellipse using the Stokes parameters, given that

$$Q = I\cos 2\chi \cos 2\psi,\ U = I \cos 2\chi \sin 2\psi,\ V = I \sin 2\chi \qquad (25)$$

where $\tan\chi = \pm b/a$ is the axial ratio of the polarization ellipse and ψ is the tilt angle relative to the x axis.

Note I and V do not depend on the orientation of the ellipse (ψ) but Q and U do, and therefore depend on coordinate system.

III. Natural light

- The previous descriptions were for a coherent field, i.e. a single frequency field with a definite phase
- A more realistic description would incorporate the fact that all fields actually exist in a finite bandwidth $d\omega$
- In this case, our Stokes parameters become functions of time and do not remain constant: like a "beat" frequency
- We can still talk about time averaged Stokes vectors however:

$$\begin{aligned} I &= <a_1^2> + <a_2^2> = <|E_1|^2> + <|E_2|^2> && (26)\\ Q &= <a_1^2> - <a_2^2> = <|E_1|^2> - <|E_2|^2> && (27)\\ U &= 2 <a_1 a_2 \cos\delta> = 2\mathrm{Re}\left\{<E_1 E_2 *>\right\} && (28)\\ V &= 2 <a_1 a_2 \sin\delta> = 2\mathrm{Im}\left\{<E_1 E_2 *>\right\} && (29) \end{aligned}$$

where the $<\cdot>$ notation here refers to an average over time.

- Since we no longer have a pure field, the relationship between Stokes parameters is modified:

$$I^2 \geq Q^2 + U^2 + V^2 \qquad (30)$$

- "Natural light" is defined to have the same intensity in all polarizations perpendicular to the direction of propagation
- Note this translates into no correlation between rectangular field components: if there were a correlation, there would be a distinct polarization
- Therefore for natural light:

$$\begin{aligned} I &= <|E_1|^2> + <|E_2|^2> = 2 <|E|^2> && (31)\\ Q &= U = V = 0 && (32) \end{aligned}$$

- We can define a "degree of polarization", m, through

$$m = \frac{\sqrt{Q^2 + U^2 + V^2}}{I} \qquad (33)$$

which should be 1 for a single frequency field, 0 for natural light, and somewhere in-between for a "partial" polarization

- Note if we have two fields present, it is easy to represent the total electric fields as the sum of each
- However it is NOT easy to represent the total Stokes vector since the Stokes vector is expressed in terms of several power-like (i.e. non-linear) quantities
- If fields are first found, a total Stokes vector can then be computed from the fields
- It will be of interest in many problems to claim that Stokes vectors can be added to produce the Stokes vector of the total field, but this must always be justified
- One case where it will be claimed: completely independent particles: no phase relationships: add Stokes vectors

IV. Polarized scattering

- Now we are ready to talk about general polarized scattering
- Let's first simplify the geometry: incident field propagates in z direction, yz plane contains incident and scattering directions
- We can resolve the incident field polarization into a perpendicular and parallel component to the yz plane
- Define a scattering amplitude matrix through

$$\begin{pmatrix} E_{s\perp} \\ E_{s\parallel} \end{pmatrix} = \left(\frac{e^{ikR}}{R}\right) \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} E_{i\perp} \\ E_{i\parallel} \end{pmatrix} \qquad (34)$$

- Note here the incident field amplitudes are evaluated at the origin, while the scattered fields are in the far field
- Note the f_{ij} quantities depend on incidence and scattering angles, frequency, etc.; rewrite angular dependence as θ

- f_{11} and f_{22} are "co-polarized" amplitudes, while f_{12} and f_{21} are "cross polarized"
- The forward scattering theorem also has to be split depending on polarization:

$$\sigma_t = \frac{4\pi}{k}\mathrm{Im}\left\{f_{11}(\theta=0)\right\} \text{ or } \frac{4\pi}{k}\mathrm{Im}\left\{f_{22}(\theta=0)\right\} \qquad (35)$$

- For spherical scatterers, there are never any cross polarized fields in this coordinate system, so $f_{12} = f_{21} = 0$
- It can be shown for backscattering that $f_{12} = f_{21}$ through use of reciprocity; this is not true however at other angles

EE 816 - Lecture 6

1. Stokes matrix
2. Particle size distribution
3. Characteristics of media
4. Atmospheric particles

I. Stokes matrix

We can also derive a relationship between incident and scattered modified Stokes parameters. Result is:

$$\overline{I}_s = \left(\frac{1}{R^2}\right)\overline{\overline{\sigma}}\,\overline{I}_i \qquad (36)$$

where

$$\overline{I}_s = \begin{bmatrix} I_{1s} \\ I_{2s} \\ U_s \\ V_s \end{bmatrix}, \ \overline{I}_i = \begin{bmatrix} I_{1i} \\ I_{2i} \\ U_i \\ V_i \end{bmatrix} \qquad (37)$$

and the "Stokes" matrix is

$$\overline{\overline{\sigma}} = \begin{bmatrix} |f_{11}|^2 & |f_{12}|^2 & \mathrm{Re}\{f_{11}f_{12}*\} & -\mathrm{Im}\{f_{11}f_{12}*\} \\ |f_{21}|^2 & |f_{22}|^2 & \mathrm{Re}\{f_{21}f_{22}*\} & -\mathrm{Im}\{f_{21}f_{22}*\} \\ 2\mathrm{Re}\{f_{11}f_{21}*\} & 2\mathrm{Re}\{f_{12}f_{22}*\} & \mathrm{Re}\{f_{11}f_{22}*+f_{12}f_{21}*\} & -\mathrm{Im}\{f_{11}f_{22}*-f_{12}f_{21} \\ 2\mathrm{Im}\{f_{11}f_{21}*\} & 2\mathrm{Im}\{f_{12}f_{22}*\} & \mathrm{Im}\{f_{11}f_{22}*+f_{12}f_{21}*\} & \mathrm{Re}\{f_{11}f_{22}*-f_{12}f_{21}* \end{bmatrix} \qquad (38)$$

- Note for spherical scatterers in this coordinate system, the Stokes matrix involves only four quantities: $|f_{11}|^2$, $|f_{22}|^2$, $\mathrm{Re}\{f_{11}f_{22}*\}$, and $\mathrm{Im}\{f_{11}f_{22}*\}$
- Ishimaru also discusses how Stokes vectors transform under a rotation: rotate definition of x and y
- He shows that the Stokes vectors in the new coordinate system are expressible in terms of those in the old coordinate system through a matrix multiplication
- Note that V remains invariant with respect to a rotation of coordinate system; I_1 and I_2 do not, but $I = I_1 + I_2$ remains invariant also

- Although Ishimaru has provided a consistent way of discussing polarized scattering, his approach is difficult to work with
- This is because the incident polarization vector definition varies as the scattered angle changes; not a typical definition.
- In this case it is easier to define global incident and scattered propagation and polarization directions
- Define as "horizontal" and "vertical" polarizations: if z direction is considered "up", define "horizontal" polarization to be ⊥ to the direciton of propagation and to z; "vertical" is then whatever is left over
- The handout "Phase matrix for Rayleight Scattering" discusses this coordinate system and also derives the Stokes and "phase" matrices in terms of a scattering matrix relating incident and scattered horizontal and vertical polarizations
- The phase matrix for Rayleigh scattering is derived in this system; will be important later on!

II. Particle size distribution

- Thus far we've been talking about deterministic single particle scattering problems
- We can imagine a statistical problem in which the particle size is a RV and we wish to compute the average RCS
- There are many ways of providing a pdf for particle sizes: begin with ρ, the total number of particles per unit volume
- We can divide ρ in terms of particle size D through

  $$\rho = \int_0^\infty n(D)dD \qquad (39)$$

  where $n(D)$ is then the number of particles per unit volume of a certain size D

- We can also define $W(D) = n(D)/\rho$, the pdf of finding a particle of size D
- Given this pdf we can compute averages of the total cross section, average size D, etc.

$$\begin{aligned} < \sigma_t > &= \int_0^\infty \sigma_t(D) W(D) dD \quad (40) \\ < D > &= \int_0^\infty D W(D) dD \quad (41) \end{aligned}$$

- These expressions are exact only for the case in which we have scattering from a single particle whose size is unknown
- If we have a large number of particles (as suggested by ρ) of different sizes, we must make additional approximations to calculate average cross sections
- We'll be making these approximations later on, so Ishimaru is jumping ahead slightly

III. Characteristics of media

- Chapter 3 of Ishimaru considers the physical characteristics of several different environments: atmosphere, underwater, biological
- Clearly these are all media that will produce scattering since they have inhomogeneities
- It is also clear that the inhomogeneities are best described statistically
- For these media we have the basic ingredients of a random medium problem!
- Note however that we require a description of the physical medium; we should not expect to obtain better predictions from any models than the accuracy of the medium description we put in
- Could also consider: forests, grasslands, sea surface, etc.

- The relationship between the EM quantities you wish to study and the description of the medium needed should be thought about before beginning any problem
- If problems involving many scatterers are considered, a complete description of the medium is a stochastic process
- Some approximations simplify things considerably: assume particle positions are independent random variables: joint pdf decouples into produce of all individual pdf's
- A stochastic process description is applied for rough surfaces: some scattering theories require only moments of process and not full joint pdf
- Again these descriptions need to match reality to some degree or else we will not get good predictions
- Note extinction cross section determines attenuation: we will study later

IV. Atmospheric particles

- Most of Ishimaru's examples are centered around scattering from the atmosphere, so we should be familiar with the problem
- Important scatterers in the atmosphere: aerosols (particles usually under 1 μm radius) and hydrometeors (water particles usually over 1 μm radius)
- Important hydrometeors: rain, fog, clouds, etc.
- Properties of rain have been studied extensively; usually expressed in terms of p, the rain rate in mm/hr
- The Marshall-Palmer distribution provides an empirical description of the rain particle size distribution
- Below 10 GHz or so, rain drops can be considered Rayleigh scatterers, and an average cross section can be computed over the size distribution
- Mie theory can be used at higher frequencies

- Ishimaru plots the average extinction and backscatter cross sections for rain and clouds, as well as the albedo
- Understanding the details of atmospheric structure can be critical is designing models for a tropo-spheric scatter communications link, discussed in later chapters
- Also atmospheric scattering is important as both clutter in typical radar systems or as the signal for weather radars
- Optical and acoustical scattering in the sea and biological materials are also discussed
- Interesting info about optical measurements to determine blood oxygenation
- Acoustic propagation in biological materials: ultrasound
- The theories we are studying here are fairly general and applied in many areas