

I. Independent Scattering

EE 816 - Lecture 7

1. Introduction to independent scattering
2. Average power
3. “First order” multiple scattering
4. Narrow beam result

1

- We defined the extinction cross section σ_t of a single particle to account for power lost due to scattering and absorption:

$$P_{lost} = \sigma_t S \quad (1)$$

where S is the incident Poynting flux on the particle

- If we have more than a single particle we approximate that all particles produce the same loss and obtain

$$P_{lost} = \rho \sigma_t S dV \quad (2)$$

where ρ is the number density of particles and dV is a small volume

- Actually this argument only applies for spherical scatterers; for non-spherical we have to worry about polarization coupling effects

3

- We begin our study of random medium theories with the simplest possible approximation: many spherical particles whose positions are totally random and independent
- We’ll also ignore any multiple scattering: “single scattering” approximation
- In this case, average scattered powers add everywhere except forward direction since there are no phase addition or cancellation effects on average
- We will describe media in terms of a particle number density and the scattering characteristics of single particles
- This is a “discrete” scatterer approach
- First we’ll clarify attenuation in this medium...

2

- Now consider a small region of space with cross sectional area A and length dl ; the total power incident at the front of the region is $P(0) = SA$
- The power exiting is $P(dl) = SA - \rho \sigma_t S dV$
- Rewrite as

$$P(dl) - P(0) = -\rho \sigma_t S dV = -\rho \sigma_t S A dl \quad (3)$$

$$\frac{S(dl) - S(0)}{dl} = -\rho \sigma_t S \quad (4)$$

$$\frac{dS}{dl} = -\rho \sigma_t S \quad (5)$$

- Thus solving the diff. eqn we find that $S(l) = S(0)e^{-\rho \sigma_t l}$, i.e. the power attenuates with attenuation rate $\rho \sigma_t$.
- For a particle size distribution, we simply use the average value of σ_t ; for non-constant $\rho \sigma_t$ along the path, the total loss follows $e^{-\int_0^l \rho(l) \sigma_t(l) dl}$
- The “optical distance” γ between two points is defined as $\int_0^l \rho(l) \sigma_t(l) dl$ since this determines the power attenuation

4

II. Average power

- Consider the average scattered power when a volume scattering medium is illuminated by a cw transmitter with gain $G_t(\hat{i})$; scattered power measured with an antenna with effective aperture $A_r(\hat{O})$
- On a volume dV at range R_1 from the source, incident flux is $S_i = P_t G_t(\hat{i}) / 4\pi R_1^2$
- The average scattered Poynting flux at range R_2 is $\langle S_r \rangle = \frac{\rho \langle |f(\hat{O}, \hat{i})|^2 \rangle}{R_2^2} S_i dV$
- Power received is $\langle P_r \rangle = A_r(\hat{O}) \langle S_r \rangle = \frac{\lambda^2}{4\pi} G_r(\hat{O}) \langle S_r \rangle$
- Adding up the power from each piece of volume we obtain:

$$\frac{\langle P_r \rangle}{P_t} = \int_V dV \frac{\lambda^2 G_t(\hat{i}) G_r(\hat{O})}{(4\pi)^3 R_1^2 R_2^2} \rho \langle \sigma_{bi}(\hat{O}, \hat{i}) \rangle \quad (6)$$

5

III. First order multiple scattering

- In the previous derivation we neglected the attenuation of the incident field due to particle scattering
- If we include it, we obtain the “first order” multiple scattering approximation:

$$\frac{\langle P_r \rangle}{P_t} = \int_V dV \frac{\lambda^2 G_t(\hat{i}) G_r(\hat{O})}{(4\pi)^3 R_1^2 R_2^2} \rho \langle \sigma_{bi}(\hat{O}, \hat{i}) \rangle e^{-\gamma_1 - \gamma_2}$$

where γ_1 is the optical depth from the transmitter to dV and γ_2 is the optical depth from dV to the receiver

- This is called a multiple scattering approximation since the attenuation caused by different particles is included in computing scattering from a single particle
- However it is still a very approximate theory, and applies again only for small volume fractions, etc.

7

- If we consider backscattering:

$$\frac{\langle P_r \rangle}{P_t} = \int_V dV \frac{\lambda^2 [G_t(\hat{i})]^2}{(4\pi)^3 R_1^4} \rho \langle \sigma_{bi}(-\hat{i}, \hat{i}) \rangle \quad (7)$$

- The above integrals are over all of space; however, the antenna gains will both be non-zero only over some “common volume” and the integral can be truncated to this volume
- Note we neglected any mismatch or other losses
- We also neglected attenuation in the medium; for the above equation to be reasonable, we need $\rho \sigma_t L \ll 1$, where L is the approximate “length” of the random medium
- This is a reasonable model for scattering when a medium has random spherical particles and a low fractional volume
- Can also generalize to the case of non-spherical scatterers, but we have to be more careful about polarization effects and things get more complicated

6

IV. Narrow beam equations

- Ishimaru next considers these equations when the antenna patterns can be modeled as narrow Gaussian beams
- He also approximates that the scatterer characteristics ρ and σ_t are constant throughout the common volume since the beams are narrow
- For backscattering, he finds

$$\frac{P_r}{P_t} = (2.855 \times 10^{-4}) \lambda^2 [G_t(\hat{i}_0)]^2 \theta_1 \phi_1 \int_0^\infty dR \frac{\rho \sigma_b}{R^2} e^{-2\gamma}$$

- Note typo in book eqn (4-18) page 76: $V_c = \frac{\pi R^2 \theta_1 \phi_1}{8 \log 2} dR$
- Useful equation for getting some practical numbers for random medium scattering; examples discussed include optical scattering in water and microwave backscatter from rain
- This is the basic theory used to study microwave scattering from rain

8

- Note the received power is falling off slower than the $1/R^4$ of our single particle equation
- This is because the common volume grows with range, so more scatterers are contributing
- This is a typical characteristic of atmospheric, surface, or volume clutter: amount of power received depends on the illuminated area or volume
- Therefore in many problems it makes more sense to talk about a cross section per unit volume or unit area
- Here we obtained a prediction of average powers very easily; in other theories this will be much more of a challenge
- Due to simplicity of this theory, Ishimaru goes on to study more complex questions

9

I. Coherent and incoherent power

- By coherent power, we mean a quantity proportional to $|\langle \bar{E} \rangle|^2$
- Thus a “coherent” field retains its phase even when scatterers are randomized
- With totally random positions, only exists in forward scatter direction if at all
- By incoherent power, we mean a quantity proportional to $\langle |\bar{E} - \langle \bar{E} \rangle|^2 \rangle$
- An incoherent field does not have a definite phase; varies randomly as scatterer positions vary
- Both coherent and incoherent powers are important
- With only one realization we cannot separate coherent and incoherent

11

EE 816 - Lecture 8

1. Review “coherent” and “incoherent” power
2. Moving particles: fundamental ideas
3. Time coherent and incoherent fields
4. Time correlated cross section for one particle

10

II. Moving particles: fundamental ideas

- So far we’ve been thinking about problems involving fixed particles, and considering averages over different realizations of the particle positions
- However now Ishimaru introduces another level of complexity: particles whose position changes in time
- Thus even for one initial configuration of particles we can now talk about a time series of measurements as the particles move: a stochastic process!
- This allows us to define a second ensemble average over time
- We need to be careful about our averages; distinguish between realization averaging and time averaging
- Clearly there can be some effects: moving particles cause Doppler shifts

12

III. Time coherent and incoherent fields

Since moving particles are causing Doppler shifts, i.e. frequency spreading of our CW field, let's write the received field as

$$v(\bar{r}, t) = \text{Re} \{u(\bar{r}, t) \exp(-i\omega_0 t)\} \quad (8)$$

$$u(\bar{r}, t) = A(\bar{r}, t) \exp(i\phi(\bar{r}, t)) \quad (9)$$

where u is called the “complex field”. Here the amplitude A and phase ϕ modulations are assumed to be much slower varying in time than ω_0

We can now describe u as a time average part and a “fluctuating” part:

$$u(\bar{r}, t) = \langle u(\bar{r}, t) \rangle + u_f \quad (10)$$

for which $\langle u_f \rangle = 0$

13

Consider the time covariance between fields at a single point:

$$\Gamma_f(\bar{r}_1, t_1; \bar{r}_1, t_2) = \langle u_f(\bar{r}_1, t_1) u_f^*(\bar{r}_1, t_2) \rangle \quad (17)$$

$$\Gamma_f(\bar{r}_1, \tau) = \langle u_f(\bar{r}_1, t_1) u_f^*(\bar{r}_1, t_1 + \tau) \rangle \quad (18)$$

where the second line applies to a random process that is “stationary” in time, i.e. whose statistical properties do not change in time. (Usually reasonable over some limited time interval)

We now have the time autocorrelation function of the fluctuating field; the Fourier transform of this function is the temporal frequency spectrum:

$$W_f(\bar{r}, \omega) = 2 \int_{-\infty}^{\infty} d\tau \Gamma_f(\bar{r}, \tau) e^{i\omega\tau} \quad (19)$$

$$\Gamma_f(\bar{r}, \tau) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega W_f(\bar{r}, \omega) e^{-i\omega\tau} \quad (20)$$

Thus W_f gives us a measurement of the amount of frequency spreading of our modulated field about ω_0

Since $\Gamma_f(\bar{r}_1, \tau) = \Gamma_f^*(\bar{r}_1, -\tau)$, W_f can be shown to be positive and real

15

$\langle u \rangle$ is the time coherent part of u while u_f is the time incoherent part. Thus we can talk about coherent and incoherent intensities as

$$\langle I \rangle = \langle |u|^2 \rangle = I_c + I_f \quad (11)$$

$$I_c = |\langle u \rangle|^2 \quad (12)$$

$$I_f = \langle |u_f|^2 \rangle \quad (13)$$

We can also talk about a “correlation” function between complex fields measured at differing times and positions:

$$\Gamma(\bar{r}_1, t_1; \bar{r}_2, t_2) = \langle u(\bar{r}_1, t_1) u^*(\bar{r}_2, t_2) \rangle = \Gamma_c + \Gamma_f \quad (14)$$

$$\Gamma_c = \langle u(\bar{r}_1, t_1) \rangle \langle u^*(\bar{r}_2, t_2) \rangle \quad (15)$$

$$\Gamma_f = \langle u_f(\bar{r}_1, t_1) u_f^*(\bar{r}_2, t_2) \rangle \quad (16)$$

Γ is called the “mutual coherence function” and Γ_f the “covariance” by Ishimaru

We can also define correlations between intensities, etc.

14

- This is our first encounter with a stochastic process, and the description we are using is as complex a description as we will encounter
- Again a stochastic process is a function (here a function of time) for which each value in time is a random variable
- Thus we can discuss either time (set up particles then take a series of measurements as the particles move) or ensemble (take measurements for a large number of particle initial states) averages
- The covariance we are discussing here is a second order moment: $\Gamma_f(\bar{r}_1, \tau) = \langle u_f(\bar{r}_1, t_1) u_f^*(\bar{r}_1, t_1 + \tau) \rangle$ involving fields at a single point but displaced in time
- Since the process is assumed stationary, we get the same answer no matter what our starting point in time is
- It is fairly straight forward to show that this correlation function is related to the temporal frequency spectrum through a Fourier transform

16

- Definition of temporal frequency spectrum: record fields as a function of time as particles move; take Fourier transform of time function to get a frequency spectrum and take magnitude squared; average over many realizations
- Clearly this measures the average frequency content of a signal
- Again this is only a second order moment description of a stochastic process; only a limited amount of information
- However in this problem it turns out that time averaged quantities involve only the second moment of the stochastic process
- Later we will consider stochastic processes as functions of space to describe rough surfaces; again second order moments will sometimes be found sufficient
- Even in this problem we can also consider scattered fields to be a stochastic function of space; we can then discuss time, space, and ensemble averages

17

Thus $\bar{r}_1 - \bar{r}_2 = \bar{V}\tau$, where $\tau = t_1 - t_2$. Define a time correlated cross section as

$$\sigma_d(\hat{O}, \hat{i}, \tau) = \lim_{R_0 \rightarrow \infty} R_0^2 \langle E_s(t_1) E_s^*(t_2) \rangle \quad (23)$$

$$= \sigma_d(\hat{O}, \hat{i}) \langle \exp(i\bar{k}_s \cdot \bar{V}\tau) \rangle \quad (24)$$

where $\bar{k}_s = k(\hat{i} - \hat{O})$ and $\sigma_d(\hat{O}, \hat{i})$ is the non-moving particle cross section. Note the time correlated version can be complex.

Note $\langle \exp(i\bar{k}_s \cdot \bar{V}\tau) \rangle$ is a characteristic function for the \bar{V} random variable. If we separate $\bar{V} = \langle \bar{V} \rangle + \bar{V}_f = \bar{U} + \bar{V}_f$ then

$$\sigma_d(\hat{O}, \hat{i}, \tau) = \sigma_d(\hat{O}, \hat{i}) \chi(\bar{k}_s \tau) \exp(i\bar{k}_s \cdot \bar{U}\tau) \quad (25)$$

thus we have obtained the time correlated cross section of a single particle in terms of the statistical properties of its velocity

Ishimaru also discusses a rotational average for rotating (non-spherical) particles

19

III. Time correlated cross section for one particle

Consider a slowly moving particle with velocity \bar{V} , illuminated by a plane wave $\exp(k\bar{r} \cdot \hat{i})$. The scattered field at distance R_0 and time t_1 is given by

$$E_s(t_1) = \frac{f(\bar{O}, \bar{i})}{R_0} \exp(ik(\hat{i} - \hat{O}) \cdot \bar{r}_1 + ikR_0) \quad (21)$$

where \bar{r}_1 is the location of the particle at time $t'_1 = t_1 - R_1/c$.

At time t_2 the scattered field is

$$E_s(t_2) = \frac{f(\bar{O}, \bar{i})}{R_0} \exp(ik(\hat{i} - \hat{O}) \cdot \bar{r}_2 + ikR_0) \quad (22)$$

where \bar{r}_2 is the location of the particle at time $t'_2 = t_2 - R_2/c$.

Now $\bar{r}_1 - \bar{r}_2 = \int_{t'_2}^{t'_1} dt \bar{V}(t)$ since the velocity in general can vary in time. If we approximate the velocity as constant, $\bar{r}_1 - \bar{r}_2 = (t'_1 - t'_2)\bar{V} \approx (t_1 - t_2)\bar{V}$ for non-relativistic particles

18

Given that we now have a field time correlation function, we can compute the temporal frequency spectrum as

$$W_\sigma(\hat{O}, \hat{i}, \omega) = 2 \int_{-\infty}^{\infty} d\tau \sigma_d(\hat{O}, \hat{i}, \tau) e^{i\omega\tau} \quad (26)$$

which provides a measure of the frequency spreading of the scattered fields

Example: non-rotating particle moving with a constant velocity \bar{U} . Since $\bar{V}_f = 0$, we have

$$\sigma_d(\hat{O}, \hat{i}, \tau) = \sigma_d(\hat{O}, \hat{i}) \exp(i\bar{k}_s \cdot \bar{U}\tau) \quad (27)$$

$$W_\sigma(\hat{O}, \hat{i}, \omega) = 2\sigma_d(\hat{O}, \hat{i}) \delta(\omega + \bar{k}_s \cdot \bar{U}) \quad (28)$$

Plugging in $\bar{k}_s = \frac{\omega_0}{c}(\hat{i} - \hat{O})$ we find a Doppler shift of

$$\frac{\omega}{\omega_0} = -(\hat{i} - \hat{O}) \cdot \bar{U}/c \quad (29)$$

$$\omega + \omega_0 = \omega_0 \left[1 - (\hat{i} - \hat{O}) \cdot \bar{U}/c \right] \quad (30)$$

which can be rewritten as $\omega/\omega_0 = -2 \sin(\theta/2)(U_d/c)$, where U_d is the component of \bar{U} along the direction $\hat{i} - \hat{O}$

20

Another example: particle with average velocity \bar{U} plus a normally (Gaussian) distributed velocity fluctuation in each component with variance σ_f^2 , then

$$p(\bar{V}_f) = (2\pi\sigma_f^2)^{-3/2} \exp(-|\bar{V}_f|^2/2\sigma_f^2) \quad (31)$$

$$\chi(\bar{k}_s\tau) = \exp(-|\bar{k}_s|^2\sigma_f^2\tau^2/2) \quad (32)$$

and it can be shown that

$$W_\sigma(\hat{O}, \hat{i}, \omega) = 2\sigma_d(\hat{O}, \hat{i}) \sqrt{\frac{2\pi}{|\bar{k}_s|^2\sigma_f^2}} \exp\left(-\frac{(\omega + \bar{k}_s \cdot \bar{U})^2}{2|\bar{k}_s|^2\sigma_f^2}\right) \quad (33)$$

which shows that a Gaussian spread in frequency is obtained about the Doppler frequency associated with \bar{U} instead of the previous delta function.

This illustrates “Doppler broadening” due to a randomness in particle velocity

21

I. Scattered field time correlation function

Having considered the one particle case, we can now generalize to many particles; again this is easy since we are assuming independent scattering. We now talk about a correlation function $B_v(\tau)$ for the voltage $V(t)$ in the receiver as

$$B_v(\tau) = \langle V(t_1)V^*(t_2) \rangle = c \int_V dV \frac{\lambda^2 G_t(\hat{i}) G_r(\hat{O})}{(4\pi)^3 R_1^2 R_2^2} \rho < \sigma_{bi}(\hat{O}, \hat{i}, \tau) \rangle e^{-\gamma_1 - \gamma_2}$$

where $\langle \sigma_{bi}(\hat{O}, \hat{i}, \tau) \rangle = 4\pi \langle \sigma_d(\hat{O}, \hat{i}, \tau) \rangle$, given by our previous result

We can also compute the temporal frequency spectrum $W_v(\omega)$ by taking the Fourier transform of this. It turns out we get just the same equation as above but replace σ_{bi} with $4\pi W_\sigma(\hat{O}, \hat{i}, \omega)$ that we found previously.

23

EE 816 - Lecture 9

1. Scattered field time correlation function
2. Spatial correlation functions
3. PDF of scattered fields

22

Ishimaru next considers the case when it is known that particles of differing sizes have differing velocities (for example raindrops). In this case, the time correlated cross section averaged over particle size must include these effects:

$$\rho < \sigma_{bi}(\hat{O}, \hat{i}, \tau) \rangle = \int_0^\infty dD n(D) < \sigma_{bi}(\hat{O}, \hat{i}, \tau, D) \rangle \quad (34)$$

The example (for deterministic velocity particles) shows that a drop size distribution can be recovered from Doppler spectrum measurements

Notice we’re getting more and more random dimensions: particle positions, particle sizes, time variations, but these are still manageable in an independent scattering theory

It seems reasonable that time and ensemble average powers should eventually become identical - an “ergodic” process

Our equations can also apply to the case of fixed particles and a moving receiver with some small modifications

24

II. Spatial correlation functions

- We can also talk about a correlation function between fields at the same time measured at two points in space
- This is important for “diversity” type receivers
- Consider two receivers $P_1 = (d/2, 0, 0)$ and $P_2 = (-d/2, 0, 0)$. Our derivation for received powers still applies as before except now an additional phase shift is introduced
- In the far field, the correlation function will contain a phase factor $e^{i\psi} = \exp(-kd\hat{O} \cdot \hat{x})$
- The spatial correlation between voltages in the two receivers is then given by

$$\langle V_1 V_2^* \rangle = c \int_V dV \frac{\lambda^2 G_t(\hat{i}) G_r(\hat{O})}{(4\pi)^3 R_1^2 R_2^2} \rho \langle \sigma_{bi}(\hat{O}, \hat{i}) \rangle e^{-\gamma_1 - \gamma_2 + i\psi}$$

25

III. PDF of scattered fields

- In the independent scattering theory, our total scattered field is the sum of similar amplitude scattered fields from a large number of particles
- Individual contributions are also independent since we have no multiple scattering effects
- It can be shown that the pdf for a RV defined as the sum of a large number of independent RV's approaches to a Gaussian pdf
- This is the “Central limit theorem”
- Thus we should expect both the real and imaginary parts of the total field to be Gaussian distributed; both are given by a sum of many RV's
- Also the real and imaginary parts are independent as well since all the positions are random

27

- If we consider a transmitter located at the origin:

$$\langle V_1 V_2^* \rangle = c \int_V dV \frac{\lambda^2 G_t(\hat{i}) G_r(-\hat{i})}{(4\pi)^3 R^4} \rho \langle \sigma_{bi}(-\hat{i}, \hat{i}) \rangle e^{-\gamma_1 - \gamma_2 + ikd(\hat{i} \cdot \hat{x})}$$

- Note as we integrate over volume, \hat{i} is changing: this is more or less a Fourier transform of the antenna gain pattern evaluated at something like kd
- Wide antenna beams = narrow correlation function, also larger d = lower correlation
- Ishimaru considers an example involving a narrow volume scattering layer

26

- It is clear the phase of the total field should be completely random, i.e. uniformly distributed from 0 to 2π
- The amplitude of the field should have the pdf of $\sqrt{X^2 + Y^2}$ where X and Y are the field real and imaginary parts and are independent Gaussian RV's
- This pdf can be shown to Rayleigh; thus the amplitude of scattered fields has a Rayleigh distribution
- The Rayleigh distribution is the expected pdf of scattered field amplitudes in random medium problems in almost all cases
- Non-Rayleigh pdf's indicate that something is not totally random or that multiple scattering effects are becoming important
- Studies of measured pdf's can give insight into the accuracy of a given random medium model

28