EE 816 - Lecture 16

1. Plane parallel problem: discretized angles solution
2. RT equations
3. Possible phase functions
4. Gaussian quadrature

I. Discretized angles solution

- We have now studied the first order iterative solution for a plane parallel medium
- However it is limited to the tenuous medium case, i.e. small scattering effects
- RT theory itself is not limited to this case; try to find a better solution
- Our next solution method discretizes the scattering integral into a sum; resulting equation can be solved using eigenvalue techniques
- The discretization is done using Gaussian quadrature; standard method for approximating an integral, analogous to Simpson's rule but more accurate
- Essentially a numerical solution of the RT equation, but some analytical forms can still be obtained and studied for a small number of points in the quadrature

II. RT equations: plane parallel medium

Start again with

$$\frac{dI(\bar{r},\hat{s})}{ds} = -\rho\sigma_t\, I(\bar{r},\hat{s}) + \frac{\rho\sigma_t}{4\pi}\int_{4\pi} d\omega'\; p(\hat{s},\hat{s}')I(\bar{r},\hat{s}')$$

but consider only normal incidence (no azimuth dependency) and neglect polarization effects; also separate into reduced incident and diffuse components as

$$I_{ri}(\tau,\mu) = F_0 e^{-\tau}\delta(\mu-1)\delta(\phi) \quad (1)$$

$$\mu\frac{dI_d(\tau,\mu)}{d\tau} = -I_d(\tau,\mu) + \frac{1}{4\pi}\int_{-1}^{1} d\mu' \int_0^{2\pi} d\phi'\; p(\mu,\phi,\mu',\phi')I_d(\tau,\mu') + \frac{p(\mu,\phi,1,0)}{4\pi}F_0 e^{-\tau}$$

where $\mu = \cos\theta$ and we have assumed a normally incident plane wave

Note since for normal incidence I is independent of ϕ we can go ahead and perform the integrals over ϕ'. Also we can integrate the entire equation over ϕ to obtain:

$$\mu\frac{dI_d(\tau,\mu)}{d\tau} = -I_d(\tau,\mu) + \frac{1}{2}\int_{-1}^{1} d\mu'\; p_0(\mu,\mu')I_d(\tau,\mu') + \frac{p_0(\mu,1)}{4\pi}F_0 e^{-\tau}$$

where

$$p_0(\mu,\mu') = \frac{1}{2\pi}\int_0^{2\pi} d\phi\; \frac{1}{2\pi}\int_0^{2\pi} d\phi' p(\mu,\phi,\mu',\phi') \quad (2)$$

Also the boundary conditions in this problem remain

$$I_d(0,\mu) = 0 \quad 0 \le \mu \le 1 \quad (3)$$

$$I_d(\tau_0,\mu) = 0 \quad -1 \le \mu \le 0 \quad (4)$$

neglecting reflections at any boundaries

III. Possible phase functions

- Before getting into the numerical solution for I_d Ishimaru reviews some typical phase functions and their properties
- Common choices:
  - Isotropic: $p_0(\mu, \mu') = \frac{\sigma_s}{\sigma_t} = W_0$
  - Rayleigh: $p_0(\mu, \mu') = \frac{3}{4}\left[1 + \mu^2\mu'^2 + \frac{1}{2}(1-\mu^2)(1-\mu'^2)\right]$
  - General: expand in terms of Legendre polynomials
- Note these are already integrated over ϕ and ϕ'
- All of these functions turn out to have some symmetry properties that will be useful later on:

$$p_0(\mu, \mu') = p_0(\mu', \mu) = p_0(-\mu, -\mu') = p_0(-\mu', -\mu) \quad (5)$$

IV. Gaussian quadrature

- The basic idea of the numerical solution is to discretize the scattering integral in the RT equation:

$$\frac{1}{2}\int_{-1}^{1} d\mu' \; p_0(\mu, \mu') I_d(\tau, \mu') \quad (6)$$

  into a sum

$$\frac{1}{2}\int_{-1}^{1} d\mu' \; p_0(\mu, \mu') I_d(\tau, \mu') \approx \frac{1}{2}\sum_{j=-N}^{N} a_j \; p_0(\mu, \mu_j) I_d(\tau, \mu_j) \quad (7)$$

- If this can be done, and we consider only discrete angles, the RT equation becomes an eigenvalue equation
- Gaussian quadrature is a method for approximating an integral with a sum; other methods could also be used but Gaussian quadrature is typically a very accurate method
- Discretizing the integral in the RT equation means we are using discrete values of *angle*; we still have continuous functions in z or τ

Ishimaru reviews the basic ideas of Gaussian quadrature: begin by writing

$$\int_a^b f(x)dx \approx \sum_{j=1}^{m} a_j f(x_j) \quad (8)$$

here we are not going to necessarily choose x_j as evenly spaced points however.

Use the Lagrange interpolation formula to approximate $f(x)$:

$$f(x) = \sum_{j=1}^{m} f(x_j) \frac{F(x)}{(x - x_j)F'(x_j)} \quad (9)$$

where $F(x) = (x - x_1)(x - x_2)...(x - x_m)$, and the x_i's are some set of abscissas.

The above equation is exact if $f(x)$ is a polynomial of degree $m - 1$ but only approximate otherwise. The equation works because it turns out that $\frac{F(x_i)}{(x_i - x_j)F'(x_j)}$ is 1 for $x_i = x_j$ but zero if $x_i \neq x_j$

Using this representation of $f(x)$ we can derive what the coefficients a_j should be in the approximation of the integral:

$$a_j = \frac{1}{F'(x_j)} \int_a^b \frac{F(x)}{(x - x_j)} dx \quad (10)$$

again which will provide an exact answer for the integral of a polynomial of degree $m - 1$

Our x_j's are still arbitrary though; it turns out that if we choose them in a smart way we can obtain an exact answer for a polynomial of up to degree $2m - 1$!

The trick is to choose the x_j's so that $F(x)$ becomes a Legendre polynomial which has the orthogonality properties we need. In the end we need $F(x) = P_m(x)$ and therefore the x_j's are the zeros of the Legendre polynomial of order m

These are usually tabled for a given order m; Ishimaru provides the cases with $m = 2$ and $m = 4$; numerical recipes has a routine for higher order

- The integral we are interested in is

$$\frac{1}{2}\sum_{j=-N}^{N} a_j\ p_0(\mu,\mu_j) I_d(\tau,\mu_j) \tag{11}$$

- Therefore our μ_j's will be the zeros of the Legendre Polynomial of order $2N$ (j defined to be both positive and negative)
- This choice is useful because the a_j's and μ_j's have some symmetries: $a_j = a_{-j}$ and $\mu_j = -\mu_{-j}$
- In other words we evaluate the function at both positive and negative values with the same weights when calculating the integral
- Since $\mu = \cos\theta$, μ_j we will choose to be positive and therefore represents a forward going wave; μ_{-j} is backward going
- We will define the largest positive value to be μ_N, therefore the most negative value is μ_{-N}. μ_N is therefore the closest to 0 degrees, others rotate backwards

EE 816 - Lecture 17

1. Discretized angle solution of RT equation
2. Homogeneous solution
3. Matching boundary conditions and final answer
4. Example with $N = 1$

I. Discretized angle solution

We have now discretized the integral in the RT equation to obtain

$$\mu\frac{dI_d(\tau,\mu)}{d\tau} = -I_d(\tau,\mu) + \frac{1}{2}\sum_{j=-N}^{N} a_j\ p_0(\mu,\mu') I_d(\tau,\mu_j) + \frac{p_0(\mu,1)}{4\pi}F_0 e^{-\tau}$$

Now let's evaluate this equation at $\mu = \mu_i$ (same values as μ_j's):

$$\mu_i\frac{dI_d(\tau,\mu_i)}{d\tau} = -I_d(\tau,\mu_i) + \frac{1}{2}\sum_{j=-N}^{N} a_j\ p_0(\mu_i,\mu_j) I_d(\tau,\mu_j) + \frac{p_0(\mu_i,1)}{4\pi}F_0 e^{-\tau}$$

which can be re-written as a matrix equation in terms of $I_d(\tau,\mu_i)$:

$$\frac{d\overline{I}_d(\tau)}{d\tau} + \overline{\overline{S}}\cdot\overline{I}_d(\tau) = \overline{B}e^{-\tau} \tag{12}$$

This is a standard system of first order differential equations.

- We arrange $\overline{I}_d(\tau)$ as $[I_d(\tau,\mu_i)]$ with $i = (+N, ..., +1, -1, ..., -N)$
- The $\overline{\overline{S}}$ matrix contains both the extinction term $-I_d(\tau,\mu_i)$ and the scattering terms $\frac{1}{2}\sum_{j=-N}^{N} a_j\ p_0(\mu_i,\mu_j) I_d(\tau,\mu_j)$
- The right hand side contains the incident field source term $\frac{p_0(\mu_i,1)}{4\pi}F_0 e^{-\tau}$
- Ishimaru describes these components in detail; the $\overline{\overline{S}}$ matrix has several symmetry properties due to properties of p_0
- However, $\overline{\overline{S}}$ is not symmetric
- Solving a system of first order diff. eqn's requires a particular and homogeneous solution
- Guesing a particular solution is easy: plug in $\overline{I}_p(\tau) = \overline{\alpha}e^{-\tau}$ and solve for the constant vector $\overline{\alpha} = \left(\overline{\overline{S}} - \overline{\overline{U}}\right)^{-1}\cdot\overline{B}$ where $\overline{\overline{U}}$ is the identity matrix

II. Homogeneous solution

- As with any first order DE we must also include a homogeneous (or "complementary") solution with unknown constants in order to match the boundary conditions
- In this case the homogeneous solution satisfies

$$\frac{d\overline{I}_c(\tau)}{d\tau} + \overline{\overline{S}} \cdot \overline{I}_c(\tau) = 0 \tag{13}$$

- If we look for solutions of the form:

$$\overline{I}_c(\tau) = \overline{\beta} e^{\lambda\tau} \tag{14}$$

we obtain the equation

$$\left(\lambda\overline{\overline{I}} + \overline{\overline{S}}\right) \cdot \overline{\beta} = 0 \tag{15}$$

- For a non-trivial solution the determinant of the matrix must vanish: exactly an eigenvalue problem!

- The determinant equation will produce a polynomial in λ of order $2N$; therefore we find $2N$ eigenvalues λ_n; corresponding to each is an eigenvector $\overline{\beta}_n$ which contains $2N$ elements
- Ishimaru tells us that there are actually positive and negative pairs of eigenvalues $\lambda_n = -\lambda_{-n}$ whose eigenvectors are related also
- Our complementary solution is therefore written as a sum over all the eigenfunctions with unknown coefficients:

$$\overline{I}_c(\tau) = \sum_{n=1}^{N} C_n \overline{\beta}_n e^{\lambda_n \tau} + \sum_{n=1}^{N} C_{-n} \overline{\beta}_{-n} e^{-\lambda_n \tau} \tag{16}$$

- Our complete solution is thus

$$\begin{aligned} \overline{I}_d(\tau) &= \overline{I}_p(\tau) + \overline{I}_c(\tau) & (17) \\ &= \overline{\alpha} e^{-\tau} + \sum_{n=1}^{N} C_n \overline{\beta}_n e^{\lambda_n \tau} + \sum_{n=1}^{N} C_{-n} \overline{\beta}_{-n} e^{-\lambda_n \tau} & (18) \end{aligned}$$

- Discrete functions of μ but continuous functions of τ!

III. Matching BC's and final answer

- Next the boundary conditions:

$$\begin{aligned} I_d(0, \mu_i) &= 0 \quad 0 \le \mu_i \le 1 & (19) \\ I_d(\tau_0, \mu_i) &= 0 \quad -1 \le \mu_i \le 0 & (20) \end{aligned}$$

are applied to determine the unknown constants

- Writing these equations produces a system of linear equations in C_n which is solved by matrix inversion. Size of matrix = $2N$
- Once the constants are found we know $I(\tau, \mu_i)$ everywhere inside our layer; however it is discretized in angle
- Ishimaru converts back into a continuous function of angle using the RT integral equation; this integrates over angle and gets rid of discretization
- On page 211-212 Ishimaru presents the final continuous solution in terms of $\overline{\alpha}$, sums over the constants C_n, the eigenvalues λ_n and the eigenvectors $\overline{\beta}_n$

IV. $N = 1$ example

- Ishimaru illustrates the whole procedure on page 213-215 for isotropic scattering and using $N = 1$
- Since with $N = 1$ there are only two angles to consider he manages to compute things analytically; larger N's would require a computer however
- Particular solution, eigenvalues, eigenvectors all found and expressed in terms of albedo W_0
- System of equations to determine the C_n's is also simple since it is only 2 by 2; larger N's require matrix inversion routines
- Final analytical answer is interesting; shows additional terms beyond first order iterative solution
- However, care must be exercised in using this solution since it was based only on $N = 1$; larger values of N should give better answers but would not be easy to write analytically

Review of procedure:

- Choose an order of quadrature N; look up values of a_j's and μ_j's
- Compute phase matrix using $p_0(\mu_i, \mu_j)$, this will be a $2N$ by $2N$ matrix
- Construct the $\overline{\overline{S}}$ matrix from phase matrix and extinction term as given in book; should have correct symmetry properties
- Construct the right hand side vector $\overline{B}$ as specified in book
- Determine particular solution $\overline{I}_p(\tau) = \overline{\alpha} e^{-\tau}$ through

$$\overline{\alpha} = \left(\overline{\overline{S}} - \overline{\overline{U}} \right)^{-1} \cdot \overline{B} \tag{21}$$

- Determine complementary solution from eigenvalues and eigenvectors of the $\overline{\overline{S}}$ matrix; canned computer routines to do this

- NOTE: canned routines must be used carefully to make sure they solve the appropriate eigenproblem; also result probably needs rearranging to match Ishimaru's ordering scheme
- Construct the boundary condition matrix equation using eigenvalues and eigenvectors; invert to determine C_n's
- Once this is completed, the solution for any value of τ or μ can be computed using Ishimaru's equations p. 211-212; requires a sum over N to do so
- To obtain backscattering, find $I_d(0, -1)$ (i.e. intensity at $\tau = 0$ going in $-z$ direction). A backscattering coefficient σ is found from 4π times this
- Note our method only works for normal incidence: for oblique incidence we have to worry about variations in ϕ. Can still be done as mentioned in $11 - 6$ by decomposing things into azimuthal harmonics