EE 816 - Lecture 18

1. Continuous medium theory
2. Received power and RCS per unit volume
3. Born approximation
4. First order results

I. Continuous medium theory

- We now move to a new theory of scattering from a random medium: continuous medium theory
- Models medium as having continuous, not discrete, variations in ϵ
- We will solve using the Born approximation: compute scattering assuming fields equal incident field
- Can iterate the Born approximation to improve, but we will be satisfied with first order
- Answer will depend on statistics of ϵ: model as a stochastic process $\epsilon(\overline{r})$ or even $\epsilon(\overline{r}, t)$
- Goal is an average RCS per unit volume

II. Received power and RCS per unit volume

- Consider a random medium illuminated by a transmitter with gain $G_t(\hat{i})$
- For a small volume δV inside the random medium, assume illuminating field is same as incident field (weak scattering case)
- Describe scattering process in terms of a differential scattering cross section per unit volume:

$$\frac{P_r}{P_t} = \frac{\lambda^2 G_t(\hat{i}) G_r(\hat{O})}{(4\pi)^2 R_1^2 R_2^2} \sigma(\hat{O}, \hat{i}) \delta V \qquad (1)$$

- For this to work, δV needs to be small enough so that the illuminating field looks planar; also δV needs to large enough so that we capture all the length scales of the "random" medium
- The latter can be expressed as the dimensions of δV being larger than a correlation distance in the random medium

- If this is the case, scattering from adjacent volumes is uncorrelated and we can write

$$\frac{P_r}{P_t} = \int \frac{\lambda^2 G_t(\hat{i}) G_r(\hat{O})}{(4\pi)^2 R_1^2 R_2^2} \sigma(\hat{O}, \hat{i}) \, dV \qquad (2)$$

- Thus we can find our average received powers (which will be incoherent away from forward scattering) if we know σ, the differential RCSper unit volume of the random medium
- Note again here we are being a little lax about polarization effects; this equation will work though as long as we compute σ in the proper transmit and receive polarizations for a particular $\hat{i}$ and $\hat{O}$
- Note this looks similar to our independent scattering result neglecting attenuation in the medium; clearly weakly scattering medium
- For our discrete scatterer independent scattering medium, $\sigma = \rho\sigma_1$ where σ_1 is the cross section of one particle and ρ is the number of particles per unit volume

III. Born approximation

- Now compute σ for a continuous random medium using the Born approximation
- First assume average relative permittivity is 1 so that

$$\epsilon(\overline{r}) = \epsilon_0(1+\epsilon_1(\overline{r})) \qquad (3)$$

where ϵ_1 is zero mean and describes the randomly varying permittivity

- For a plane wave incident on δV, we know from the volume equivalence theorem that

$$\overline{E}_s = \overline{f}(\hat{O},\hat{i})\frac{e^{ikR}}{R} \qquad (4)$$

$$\overline{f}(\hat{O},\hat{i}) = \frac{k^2}{4\pi}\int_{\delta V}\left\{-\hat{O}\times\hat{O}\times\overline{E}(\overline{r}')\right\}\epsilon_1(\overline{r}')\exp(-ik\overline{r}'\cdot\hat{O})\,dV'$$

- Thus we can find scattered fields from δV if we know the field at δV

- Under the Born approximation, we assume $\overline{E}(\overline{r}') = \overline{E}^{inc}(\overline{r}') = \hat{e}_i\exp(ik\hat{i}\cdot\overline{r}')$
- We can then find

$$\overline{f}(\hat{O},\hat{i}) = \hat{e}_s\sin\chi\frac{k^2}{4\pi}\int_{\delta V}\epsilon_1(\overline{r}')\exp(i\overline{k}_s\cdot\overline{r}')\,dV'$$

where $\hat{e}_s\sin\chi = -\hat{O}\times\hat{O}\times\hat{e}_i$ and $\overline{k}_s = k\left(\hat{i}-\hat{O}\right)$

- It is easy to show that the magnitude of $\overline{k}_s$ is $2k\sin(\theta/2)$ where θ is the angle between $\hat{i}$ and $\hat{O}$
- Note it is also clear from the above equation that $<\overline{f}(\hat{O},\hat{i})>=0$ since $<\epsilon_1>=0$
- To get the cross section per unit volume we need

$$\left|\overline{f}(\hat{O},\hat{i})\right|^2/\delta V \qquad (5)$$

- However since $\overline{f}$ is now random, we must talk about the average cross section per unit volume, defined as

$$\sigma(\hat{O},\hat{i}) = \frac{1}{\delta V}<\overline{f}(\hat{O},\hat{i})\cdot\overline{f}^*(\hat{O},\hat{i})> \qquad (6)$$

- Plugging our integral for $\overline{f}$ into the equation for the average cross section per unit area we find:

$$\sigma(\hat{O},\hat{i}) = \frac{k^4\sin^2\chi}{(4\pi)^2\delta V}\int_{\delta V}\int_{\delta V'}<\epsilon_1(\overline{r}_1')\epsilon_1^*(\overline{r}_2')>\exp\left[i\overline{k}_s\cdot(\overline{r}_1'-\overline{r}_2')\right]dV_1'dV_2'$$

- Note in the above that we maintain the product of two integrals: cannot reduce to one integral! Be careful about this in general.
- The above expression is in terms of the correlation function $<\epsilon_1(\overline{r}_1')\epsilon_1^*(\overline{r}_2')>$ of our random permittivity fluctuations
- If we have a stationary, isotropic random process, the above depends only on the distance between $\overline{r}_1'$ and $\overline{r}_2'$

- Thus we can write

$$<\epsilon_1(\overline{r}_1')\epsilon_1^*(\overline{r}_2')> = B_\epsilon(|\overline{r}_1'-\overline{r}_2'|) = 4B_n(r_d) \qquad (7)$$

where B_n is the correlation function of n, the index of refraction (assuming small permittivity variations)

- Now work on the integral

$$\int_{\delta V}\int_{\delta V'}B_n(r_d)\exp\left[i\overline{k}_s\cdot(\overline{r}_d)\right]dV_1'dV_2'$$

- The terms in the integral are all only functions of the difference $\overline{r}_1'-\overline{r}_2'$; for this reason re-write integral in terms of $\overline{r}_d = \overline{r}_1'-\overline{r}_2'$ and $\overline{r}_c = \frac{1}{2}(\overline{r}_1'+\overline{r}_2')$
- The Jacobian of this transformation is 1; integration region is a little funny
- To simplify, recall that we defined δV to be bigger than the correlation distance of the medium; thus outside δV, B_n should be near zero. Therefore replace limits on r_d with ∞:

$$\int_{\delta V}dV_c\int_{\infty}dV_d\,B_n(r_d)\exp\left[i\overline{k}_s\cdot(\overline{r}_d)\right]$$

IV. First order results

- Integral over V_c remains only over δV due to our definition
- The integral over V_d is exactly now a Fourier transform of the correlation function; this should be the spectral density of $n(\overline{r})$:

$$\Psi_n(\overline{K}) = \frac{1}{(2\pi)^3}\int_{\infty} B_n(r_d)\exp\left[i\overline{k}_s\cdot(\overline{r}_d)\right]$$

- Since the integral over V_c just results in δV, we find

$$\sigma(\hat{O},\hat{i}) = 2\pi\; k^4\sin^2\chi\Psi_n(k_s) \qquad (8)$$

  where $k_s = 2k\sin(\theta/2)$
- In the end $\Psi_n(k_s)$ is a function of the magnitude of $\overline{k}_s$ only since we are considering an isotropic medium
- We now have the cross section per unit area of our random medium; from this we can calculate powers received

- We need to know the spectral density of the random medium in order to evaluate this cross section
- Note again that polarization effects were neglected; we're assuming the receiver receives all power in any polarization
- Re-deriving to include polarization effects should not be too difficult
- $\sin\chi$ pattern indicates dipole scattering effects
- Note as frequency increases we measure higher frequency components of the spectral density: finer scale features
- This is a "Bragg" scatter effect: only one spatial scale provides scattering
- We could iterate the Born approximation to get higher orders, but this rapidly becomes difficult
- Tsang-Kong-Shin consider the Born approximation in layered media and handle polarization effects more consistently

EE 816 - Lecture 19

1. Spectral models for random media
2. Booker-Gordon model
3. Gaussian model
4. Kolmogorov spectrum

I. Spectral models for random media

- We found last time that

$$\frac{P_r}{P_t} = \int \frac{\lambda^2 G_t(\hat{i})G_r(\hat{O})}{(4\pi)^2R_1^2R_2^2}\sigma(\hat{O},\hat{i})\;dV \qquad (9)$$

  where

$$\sigma(\hat{O},\hat{i}) = 2\pi\; k^4\sin^2\chi\Psi_n(k_s) \qquad (10)$$

  is the random medium cross section per unit area and Ψ_n is the spectral density of the permittivity random process
- This result was derived from the first order Born approximation and assumes a weakly scattering medium
- Most common application of this solution is to scattering and propagation in the atmosphere; for example troposcatter links, line of sight propagation, etc.
- To describe these problems we need to approximate the spectral density of the atmosphere

II. Booker-Gordon model

- The Booker-Gordon model assumes an exponential correlation function:

$$B_n(r_d) = < n_1^2 > \exp(-r_d/l) \tag{11}$$

- Note this has two parameters: a variance of the fluctuating part of the index of refraction, $< n_1^2 >$, and a correlation length l which describes the average size of inhomogeneities
- Note at the correlation length the correlation falls to $1/e$ of the value of $r_d = 0$
- Correlation length is also called the "scale of turbulence"
- To find the spectral density corresponding to this correlation function take the Fourier transform. Result is

$$\Psi_n(k_s) = \frac{< n_1^2 > l^3}{[1 + (k_s l)^2]^2} \frac{1}{\pi^2} \tag{12}$$

- The cross section is then given by

$$\sigma(\theta) = \frac{2}{\pi} \frac{k^4 l^3 \sin^2 \chi < n_1^2 >}{\left(1 + 4k^2 l^2 \sin^2(\theta/2)\right)^2} \tag{13}$$

- We can simplify if the correlation length is large or small compared to λ
- For small correlation lengths, $k_s l << 1$ so

$$\sigma(\theta) \approx \frac{2}{\pi} k^4 l^3 \sin^2 \chi < n_1^2 > \tag{14}$$

which shows a dipole scattering pattern which increases as k^4: Rayleigh scattering!

- For very large correlation lengths, $k_s l >> 1$ and

$$\sigma(\theta) = \frac{\sin^2 \chi < n_1^2 >}{8\pi l \sin^4(\theta/2)} \tag{15}$$

- This is more applicable to the atmosphere at microwave or higher frequencies; note independent of k and sharply peaked in forward direction

III. Gaussian model

- In the Gaussian model we approximate the correlation function as a Gaussian function:

$$B_n(r_d) = < n_1^2 > \exp(-r_d^2/l^2) \tag{16}$$

- Again we have only a variance and correlation length parameter
- Taking the Fourier transform the spectral density is found to also be Gaussian:

$$\Psi_n(k_s) = \frac{< n_1^2 > l^3}{8\pi\sqrt{\pi}} \exp(-\frac{(k_s l)^2}{4}) \tag{17}$$

- This gives a cross section of

$$\sigma(\theta) = \frac{< n_1^2 > k^4 l^3}{4\sqrt{\pi}} \sin^2 \chi \exp(-\frac{(k_s l)^2}{4}) \tag{18}$$

- Note the Gaussian spectrum falls off much more rapidly than the exponential with increasing frequency: less small scale variations

IV. Kolmogorov spectrum

- Both the exponential and gaussian models are nice because they are simple to understand, have only two parameters, and the spectrum is easy to compute from the correlation function
- Unfortunately reality in many instances is not well modeled by these "single length scale" descriptions
- The Kolmogorov spectrum is a spectrum derived from physical considerations of fluid turbulence; models atmospheric inhomogeneities better than either exponential or gaussian models and includes multiple length scales
- Basic idea: consider three length scale regions: outer scale L_0 (large), inner scale l_0 (small), and length scales between
- Spectrum is modeled as:

$$\Psi_n(K) = 0.033 C_n^2 (K^2 + 1/L_0^2)^{-11/6} \exp(-K^2/K_m^2) \tag{19}$$

where C_n, L_0, and $K_m = 5.92/l_0$ are parameters

- This is only valid for the middle and small scales; if we operate at high frequencies though we don't need to know the spectral density for large scales for scattering problems
- Note spectrum is specified, not correlation function. Could obtain correlation function from an inverse Fourier transform but it would not be a simple function.
- This is due to the multiple length scale nature of this process
- Ishimaru also discusses calculating the cross section per unit volume for an anisotropic random medium, i.e. one with different correlation lengths in different directions
- In this case we need $\Psi_n(\overline{k}_s)$ now to make sure directional dependence is correct
- This is the only change in the equation; can extend all our models in a similar fashion to include anisotropy

EE 816 - Lecture 20

1. Temporal fluctuations in a continuous random medium
2. Including random fluctuating velocities
3. Strong fluctuations
4. Narrow beam equation

I. Temporal fluctuations

- Again because the first order Born approximation gives fairly simple expressions, we can consider more complex problems
- Consider a time varying continuous random medium:

$$\epsilon(\overline{r},t) = \epsilon_0(1+\epsilon_1(\overline{r},t)) \tag{20}$$
$$n(\overline{r},t) = 1+n_1(\overline{r},t) \tag{21}$$

- Assume that time variations in medium are slow compared to EM wave frequency
- Also assume that random process is stationary both in time and space, with a correlation function written as

$$\begin{aligned} B_n(\overline{r},\tau) &= < n_1(\overline{r}_1+\overline{r},t+\tau)n_1(\overline{r}_1,t) > \\ &= \int\int d\overline{K}\ d\omega\ U(\overline{K},\omega)\exp(-i\overline{K}\cdot\overline{r}+i\omega\tau) \end{aligned} \tag{22}$$

which defines the four dimensional spectral density U

- We can also define a "time varying" spatial spectral density through

$$B_n(\overline{r},\tau) = \int d\overline{K}\ \Psi_n(\overline{K},\tau)\exp(-i\overline{K}\cdot\overline{r}) \tag{23}$$

or a "spatially varying" temporal spectral density through

$$B_n(\overline{r},\tau) = \int d\omega\ W_n(\overline{r},\tau)\exp(i\omega\tau) \tag{24}$$

- Let's assume that the random medium moves as a whole with velocity $\overline{V}$, i.e. the spatial pattern does not change, and the velocity $\overline{V}$ remains constant over a reasonable time compared to field variations
- Under this "locally frozen" condition, the index of refraction assumes the form

$$n_1(\overline{r},t+\tau) = n_1(\overline{r}-\overline{V}\tau,t) \tag{25}$$

i.e. the medium is moving as a whole with velocity $\overline{V}$

- The correlation function becomes

$$\begin{aligned} < n_1(\overline{r}_1, t_1) n_1(\overline{r}_2, t_2) > &= < n_1(\overline{r}_1 - \overline{V} t_1, 0) n_1(\overline{r}_2 - \overline{V} t_2, 0) > \\ &= B_n(\overline{r} - \overline{V}\tau) \qquad (26) \end{aligned}$$

where $\overline{r} = \overline{r}_1 - \overline{r}_2$ and $\tau = t_1 - t_2$

- Since we've reduced the correlation function down to a function of 3 variables instead of 4, we can re-define the spectral density through

$$B_n(\overline{r} - \overline{V}\tau) = \int d\overline{K} \Psi_n(\overline{K}) \exp(-i\overline{K} \cdot (\overline{r} - \overline{V}\tau))$$

which is just a shifted version of correlation and spectral density functions for the non-moving medium

- Therefore the time varying spatial spectral density is given by

$$\Psi_n(\overline{K}, \tau) = \Psi_n(\overline{K}) \exp(i\overline{K} \cdot \overline{V}\tau) \qquad (27)$$

for a medium moving with constant velocity $\overline{V}$

II. Including fluctuating velocities

- At this point we've found the four dimensional correlation function (and spectral density) assuming the entire medium moves with velocity $\overline{V}$
- However we can also consider a medium whose velocity is varying our derivation still applies (as long as the variations are slow compared to ω)
- Define the velocity $\overline{V}$ as an average and fluctuating part

$$\overline{V} = \overline{V}_0 + \overline{V}_{1f} \qquad (28)$$

- If we average $\Psi_n(\overline{K}, \tau)$ over these velocity fluctuations we find

$$\Psi_n(\overline{K}, \tau) = \Psi_n(\overline{K}) \exp(-i\overline{K} \cdot \overline{V}_0 \tau) \chi(\overline{K}\tau)$$

where $\chi(\overline{K}\tau)$ is the characteristic function of $\overline{V}_{1f}$

- If we assume the velocity fluctuations to be isotropic and to have a gaussian pdf with variance σ_v^2, the characteristic function is

$$\chi(\overline{K}\tau) = \exp(-\frac{1}{2}\left(|\overline{K}|^2 \sigma_v^2 \tau^2\right)) \qquad (29)$$

- Now that we can describe the time varying spatial spectral density of the random process, we can extend our previous theory to calculate the temporal correlation function of received fields:

$$\frac{1}{2\eta_0} < E_s(t+\tau) E_s^*(t) > = P_t \int \frac{\lambda^2 G_t(\hat{i}) G_r(\hat{O})}{(4\pi)^2 R_1^2 R_2^2} \sigma(\hat{O}, \hat{i}, \tau) \, dV$$

with

$$\sigma(\hat{O}, \hat{i}, \tau) = 2\pi k^4 \sin^2 \chi \Psi_n(\overline{k}_s) \exp(i\overline{k}_s \cdot \overline{V}_0 \tau) \chi(\overline{k}_s \tau) \quad (30)$$

- Be careful here about multiple χ's: one is an angle, the second, a characteristic function

- From the temporal correlation function of received fields we can take a fourier transform to determine the temporal frequency spectrum $W_s(\omega)$ of received fields:

$$\begin{aligned} W_s(\omega) &= 2\int_{-\infty}^{\infty} e^{i\omega\tau} < E_s(t+\tau) E_s^*(t) > \qquad (31) \\ &= 2\eta_0 P_t \int \frac{\lambda^2 G_t(\hat{i}) G_r(\hat{O})}{(4\pi)^2 R_1^2 R_2^2} W_\sigma(\hat{O}, \hat{i}, \omega) \, dV \end{aligned}$$

with

$$W_\sigma(\hat{O}, \hat{i}, \omega) = 2\int_{-\infty}^{\infty} e^{i\omega\tau} \sigma(\hat{O}, \hat{i}, \tau) \, d\tau \qquad (32)$$

- If we assume gaussian velocity fluctuations, this becomes

$$W_\sigma(\hat{O}, \hat{i}, \omega) = 2\pi k^4 \sin^2 \chi \Psi_n(\overline{k}_s) \frac{2\sqrt{2\pi}}{k_s \sigma_v} \exp(-\frac{\left(\omega + \overline{k}_s \cdot \overline{V}_0\right)^2}{2k_s^2 \sigma_v^2})$$

- There is a Doppler shift $\overline{k}_s \cdot \overline{V}_0$ due to the average medium motion along with a Gaussian spreading around this shift due to the velocity fluctuations

III. Strong fluctuations

- Our first order Born approximation assumed fields throughout our random medium to be the same as the incident field in calculating scattering
- We know in reality however that scattering will produce attenuation, so the field is not exactly the incident field
- Including these effects in a consistent manner becomes very difficult, but can be done: strong fluctuation theory
- We will not study strong fluctuation theory, but we can at least make a simple approximation to improve our Born result
- Ishimaru does this simply by including attenuation factors in the Born integral. Analogous to the first order multiple scattering result in independent scatter theory
- The attenuation rate is determined from dielectric loss and scattering; scattering cross section obtained by integration

IV. Narrow beam equation

- Ishimaru also derives a narrow beam equation for the continous random medium theory in the same manner as in Chapter 4
- Essentially replaces integral over volume by a constant times the common volume illuminated by the transmitting and receiving antennas
- Common volume is the same as in Chapter 4 if antenna patterns are assumed to be Gaussian functions
- Also obtains a simple expression for backscattering with a narrow beam
- Note with a pulsed radar the common volume is not determined entirely by the antenna patterns. Pulse width also determines one dimension of common volume