EE 816 - Lecture 24

1. Review perturbation solution
2. Temporal variations
3. Surface spectra
4. Composite surface theory

I. Review perturbation solution

- Last time we completed the first order SPM solution for polarized scattering from a PEC surface
- We found the "Bragg" scatter result: scattering at a particular frequency and angle depended only on one length scale of the surface
- Thus for a fixed frequency and angle, we can filter out all the other surface spatial frequencies and nothing will change!
- True only at first order, higher order SPM results couple different length scales together
- Theory based on small height relative to λ assumption and small slopes
- Coherent field is exactly that of a flat surface; clearly something wrong since these results cannot satisfy power conservation
- Coherent reflection coefficient reduced at 2nd order to fix this

- SPM derivation to higher order can be accomplished but becomes very difficult; up to 3rd order has been derived
- Can also implement a computer program to solve SPM equations to arbitrary order
- Different polarizations show differing variations with angle; vv backscattering remains constant for PEC as grazing is approached while hh goes to zero
- Interesting polarization effects for bistatic scattering also
- Spatial frequency of surface "sampled" in scattering increases as EM frequency or angle increases
- This idea has been applied to remote sense the spectrum of the ocean, especially at HF or lower frequencies
- At microwave frequencies, SPM seems questionable since ocean waves clearly can be larger than cm scales!

II. Temporal variations

- Ishimaru next considers the effect of a temporally varying surface on SPM first order results. Again, the basic idea is to calculate scattering from a "time-frozen" surface and let scattered fields evolve as surface evolves
- Should be applicable as long as surface variations are slow compared to ω
- Doing this with the SPM is particularly easy because we found even for a deterministic surface that the scattered field amplitudes were directly proportional to the surface Fourier coefficient $P(m,n)$
- For a time evolving surface, these Fourier coefficients become functions of time and therefore we obtain the field time dependence directly

- A simple linear approximation to ocean gravity wave propagation provides a time dependence of $e^{i(kx-\omega t)}$ with $\omega = \sqrt{gk}$. Here k is the spatial wavenumber of the ocean surface wave
- Thus our Fourier coefficients become something like $P(m,n)e^{i\omega_{mn}t}$ and the surface fields vary as $e^{i\omega_{mn}t}$ also!
- Eventually we find the time correlated cross section to be the same as our previous answers times this phase variation
- Result is a delta function for the temporal frequency spectrum of scattered fields. Note the frequency obtained is the ω of the Bragg wave for a particular frequency and observation angle. For backscattering, the bragg wave has $k_B = 2k \sin\theta_i$
- Thus observation of ocean scattered field doppler spectra can also provide information on how surface waves evolve. In reality things are much more complicated because ocean waves do not propagate in such a simple linear fashion

III. Surface spectra

- We have found the the SPM first order theory for scattering cross sections of a rough surface requires only that we know the surface spectral density function $W(k_{xs}-k_{xi}, k_{ys}-k_{yi})$
- Again for a random process this is related to the Fourier transform of the correlation function
- We can postulate different models for the surface correlation function, then by taking a Fourier transform find the spectral density and the SPM result
- Common guesses at a surface correlation function:
  - Gaussian: $C(\overline{\rho}) = h^2 e^{-\rho^2/l^2}$
  - Exponential: $C(\overline{\rho}) = h^2 e^{-\rho/l}$
  - Power law: $C(\overline{\rho}) = \text{FT}\{ak^{-n}\}$

  where h^2 refers to the surface height variance and l is a "correlation length" parameter

- As in our continuous random medium study, the first two model "single" length scale processes since there is only one length scale (l) considered
- The last model is analogous to the Kolmogorov spectrum, and includes surface variations on many length scales. This turns out to be a more realistic model for naturally occurring surfaces such as the ocean surface
- The model presented by Ishimaru in the book for the ocean spectrum is a power law type model.
- Note the above models have no directional dependence (i.e. functions of ρ not $\overline{\rho}$). However a realistic model for the ocean spectrum should include directional dependencies.
- The ocean spectrum is well known for fairly large length scales (say larger than 1 m), but is very difficult to measure for the small length scales (order of cm) which produce scattering at microwave frequencies. Lots of current work in this area.

IV. Composite surface theory

- Ishimaru also mentions a "composite" surface theory commonly applied to study microwave scattering from the ocean
- This theory evolved from comparisons of measured backscattering from the ocean with the first order SPM
- It turns out that first order SPM matches measured oblique vv data very well, even when ocean surface heights are very large compared to λ. However, first order SPM somewhat underestimates hh results (within 10 dB say) especially as the incidence angle increases
- These observations led to postulation of a "tilted" SPM theory to account for large waves on the ocean surface
- Idea: chop ocean spectrum into two pieces: short and long wave parts. Short wave part has small enough heights to be solved by first order SPM.

- Effect of long wave parts is simply to "tilt" the short wave parts, i.e. cause the incidence angle to be different
- In the end, first order SPM is used, but averaged over the tilt angles cause by the long wave part of the spectrum
- Tilt angles to use determined by variance of slopes of long wave surface
- This tilting procedure does not affect vv much due to its near constant value as grazing is approached. Average over angles of a constant has little effect
- However, since hh is rapidly decreasing, an average over angles tends to increase hh cross sections to be closer to measurements
- Widely accepted theory for ocean scattering at microwave frequencies. However, not derived rigorously but rather based on physical expectations
- Some more precise theories can provide a more sound basis for this approach. However, breaks down at very large angles.

- We've now completed our study of first order perturbation theory: well known theory for surface scattering and provides intuition about Bragg scatter effects
- Should be precise for small height, small slope surfaces, but turns out also to be fairly accurate for ocean surface scattering even without small heights (except near specular direction)
- Next we'll move on to study the physical optics approximation for rough surface scattering, also called Kirchhoff approach
- This method (as any physical optics method) will be most useful for near specular results and for smooth surfaces (large radius of curvature relative to λ). Can work even if surface heights are large relative to λ
- Single scattering PO solution: neglects shadowing and multiple scatter effects

EE 816 - Lecture 25

1. Kirchhoff approach: PEC surface
2. Physical optics currents
3. Scattered fields
4. Averages

I. Kirchhoff approach: PEC surface

- Now we'll consider a distinct theory for rough surface scattering: the Kirchhoff approach
- Essentially this is just applying physical optics to rough surface scattering, except shadowing effects are often not included
- Ishimaru treats an acoustic case; PEC electromagnetic case is not too difficult, so instead refer to handout for EM derivation
- As with any PO solution, results will be most accurate for near specular results; also surface should be smooth on scale of λ
- This can also be stated as "for surfaces having large radii of curvature compared to λ"
- Thus this is a high frequency theory, SPM is a low frequency theory
- Should work for near specular even if surface heights get large

- We'll overview the derivation, follow the handout for the details
- Consider a surface $z = \zeta(x,y)$ excited by an incident plane wave

$$\overline{E}^{inc} = \hat{e}_i e^{i\beta x - i\gamma z} = \hat{e}_i e^{i\overline{k}_i \cdot \overline{r}} \quad (1)$$

$$\overline{H}^{inc} = \frac{1}{\eta_0}(\hat{k}_i \times \hat{e}_i) e^{i\overline{k}_i \cdot \overline{r}} \quad (2)$$

- We'll consider $\hat{e}_i$ to be horizontally polarized. For incidence in the xz plane this gives $\hat{e}_i = \hat{y}$, with $\hat{k}_i \times \hat{e}_i = \hat{x}\cos\theta_i + \hat{z}\sin\theta_i$
- You'll work on vertical polarization in your homework!
- Again $\hat{n}$ is given by

$$\frac{1}{\sqrt{1 + (\frac{\partial\zeta}{\partial x})^2 + (\frac{\partial\zeta}{\partial y})^2}} \left(\hat{z} - \hat{x}\frac{\partial\zeta}{\partial x} - \hat{y}\frac{\partial\zeta}{\partial y} \right) \quad (3)$$

- So on our PEC surface, the physical optics induced current $\overline{J}_{PO}$ is $2\hat{n} \times \overline{H}^{inc}$

II. Physical optics currents

- Applying this and working out the cross products we find

$$\overline{J}_{PO} = \frac{2}{\eta_0} \frac{e^{ik_{ix}x} e^{-ik_{iz}z}}{\sqrt{1 + (\frac{\partial\zeta}{\partial x})^2 + (\frac{\partial\zeta}{\partial y})^2}} \left[\frac{\partial\zeta}{\partial y}(\hat{z}\cos\theta_i - \hat{x}\sin\theta_i) + \hat{y}(\cos\theta_i + \sin\theta_i \frac{\partial\zeta}{\partial x}\right.$$

  evaluated at $z = \zeta(x,y)$
- Note there is a phase variation both horizontally and vertically as we move along the surface profile
- Now we allow these currents to radiate to find the fields in the far field. This is a little easier if we first find the vector potential and then the fields
- Recall in the far field that

$$\overline{A}(\overline{r}) = \frac{\mu_0}{4\pi r} e^{ikr} \int\int dS' \ \overline{J}_{PO}(\overline{r}') e^{-ik\hat{a}_R \cdot \overline{r}'}$$

  with $\hat{a}_R = \hat{x}\sin\theta_s\cos\phi_s + \hat{y}\sin\theta_s\sin\phi_s + \hat{z}\cos\theta_s = \overline{k}_s/k$ and $\overline{r}' = \hat{x}x' + \hat{y}y' + \hat{z}\zeta(x',y')$

- Plugging in the PO currents, we'll get phase factors like $e^{ik_{ix}x'} e^{-ik_{iz}z'} e^{-ik_{sx}x'} e^{-ik_{sy}y'} e^{-ik_{sz}z'}$ which can be re-written as $e^{i\overline{k}_d \cdot \overline{r}'}$
- Also projecting the integral onto $dx'dy'$ instead of dS' through $dS' = \sqrt{1 + (\frac{\partial\zeta}{\partial x})^2 + (\frac{\partial\zeta}{\partial y})^2} dx'dy'$ we get rid of the $\sqrt{1 + (\frac{\partial\zeta}{\partial x})^2 + (\frac{\partial\zeta}{\partial y})^2}$ term in the currents
- Also note we've got these phase terms times derivative terms like $\frac{\partial\zeta}{\partial x'}$ in some cases.
- These can be simplified through an integration by parts as described in the handout. Results in an integral without the derivatives plus an "edge" term; the edge term is neglected for a large surface
- After finding the vector potential in the far field, electric fields in the far field are found from $i\omega\overline{A}_T$, where the T subscript refers to components transverse to the direction of propagation

III. Scattered fields

- Taking $\hat{h}_s$ and $\hat{v}_s$ dotted with $\overline{A}$ takes care of this for us. $\hat{h}_s$ and $\hat{v}_s$ defined as in the handout on Rayleigh scattering
- Eventually we find

$$\hat{h}_s \cdot \overline{E}_s = \frac{2ik}{4\pi r} e^{ikr} I \left[\frac{(1 + \cos\theta_i\cos\theta_s)\cos\phi_s - \sin\theta_i\sin\theta_s}{\cos\theta_i + \cos\theta_s} \right]$$

$$\hat{v}_s \cdot \overline{E}_s = \frac{2ik}{4\pi r} e^{ikr} I \left[\sin\phi_s \right]$$

- The integral I is defined as

$$\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \ e^{ik_{dx}x'} e^{ik_{dy}y'} e^{ik_{dz}\zeta(x',y')} \quad (4)$$

- The above looks like a Fourier transform
- Notice also that the cross polarized term (here $\hat{v}_s \cdot \overline{E}_s$) is zero in the plane of incidence $\phi_s = 0$

IV. Averages

- Notice that the only thing random in this problem is ζ, so if we take averages we only need to average the terms involving ζ, which here is only I
- Thus, coherent fields will go as

$$\begin{aligned} < I > &= < \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \; e^{ik_{dx}x'} e^{ik_{dy}y'} e^{ik_{dz}\zeta(x',y')} > \\ &= \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \; e^{ik_{dx}x'} e^{ik_{dy}y'} < e^{ik_{dz}\zeta(x',y')} > \end{aligned}$$

  which involves only the characteristic function of a surface point!
- Note this is at a specific point x', y' so there are no stochastic process effects involved
- Let's assume that the surface height at each point is a Gaussian random variable, i.e. $\zeta(x', y')$ is a Gaussian random variable for all x', y' with variance h^2

- This type of surface is called a Gaussian random process; it turns out that for this type of process, all statistics are completely specified by the correlation function or PSD
- Since the surface height at a point is a Gaussian random variable, we simply use the characteristic function of a GRV to evaluate $< I >$:

$$\begin{aligned} < I > &= \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \; e^{ik_{dx}x'} e^{ik_{dy}y'} < e^{ik_{dz}\zeta(x',y')} > \\ &= \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \; e^{ik_{dx}x'} e^{ik_{dy}y'} e^{-(k_{dz}h)^2/2} \\ &= (2\pi)^2 e^{-(k_{dz}h)^2/2} \delta(k_{dx})\delta(k_{dy}) \end{aligned}$$

  since we wind up taking the Fourier transform of a constant
- The delta functions indicate that $< I >$ only exists for $k_{dx} = k_{dy} = 0$, which is exactly the specular direction
- Accordingly the coherent field exists only in the specular direction, and turns out to be a plane wave reduced by the factor $e^{-(k_{dz}h)^2/2}$

- Note as roughness increases this rapidly goes to zero. SPM did not capture this effect due to the small height limit
- Now let's work on the incoherent fields $< |\hat{h}_s \cdot \overline{E}_s|^2 > - | < \hat{h}_s \cdot \overline{E}_s > |^2$ for example
- Again since I is the only thing random the statistics turn out to be taken only over I and we need $< |I|^2 > - | < I > |^2$
- Formulating $< |I|^2 >$ we obtain

$$\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dx'' \int_{-\infty}^{\infty} dy'' e^{ik_{dx}(x'-x'')} e^{ik_{dy}(y'-y'')} < e^{ik_{dz}(\zeta(x',y')-\zeta(x'',y''))} >$$

  for which we need a surface "two point" characteristic function
- Since we are assuming all the surface points are Gaussian, the joint pdf for two surface points is given by

$$f_{\zeta',\zeta''}(\zeta', \zeta'') = \frac{1}{2\pi h^2 \sqrt{1 - C^2}} \exp\left[- \left(\frac{\zeta'^2 - 2C\zeta'\zeta'' + \zeta''^2}{2h^2(1 - C^2)} \right) \right]$$

  which we found on PS#1 was the joint pdf for two Gaussian RV's with zero mean, each with variance h^2 and with correlation coefficient C

- Note C here is the correlation coefficient between the two surface heights ζ' and ζ'', which we know for a stationary, isotropic, stochastic process will depend only on the horizontal separation between points: $C = C(\sqrt{(x' - x'')^2 + (y' - y'')^2}) = C(\rho)$
- We now have described our surface in terms of its correlation function, found from $h^2 C(\rho) = < \zeta(x, y)\zeta(x + \rho, y) >$ for an isotropic surface
- Using this joint pdf, we can evaluate the two point characteristic function by integrating, and find

$$< e^{ik_{dz}(\zeta(x',y')-\zeta(x'',y''))} > = e^{-(k_{dz}h)^2(1-C(\rho))} \tag{5}$$

- Thus we now have

$$< |I|^2 > = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dx'' \int_{-\infty}^{\infty} dy'' e^{ik_{dx}(x'-x'')} e^{ik_{dy}(y'-y'')} e^{-(k_{dz}h)^2(1-C(\rho))}$$

  again with $\rho = \sqrt{(x' - x'')^2 + (y' - y'')^2}$

EE 816 - Lecture 26

1. Incoherent fields
2. Cross sections per unit area
3. Small height limit
4. Large height limit

I. Incoherent fields

- Last time we found

$$< |I|^2 > = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dx'' \int_{-\infty}^{\infty} dy'' e^{ik_{dx}(x'-x'')} e^{ik_{dy}(y'-y'')} e^{-(k_{dz}h)^2(1-C(\rho))}$$

with $\rho = \sqrt{(x'-x'')^2 + (y'-y'')^2}$

- As in our continuous random medium study, the integrand in the above four fold integral depends only on the difference coordinate, so changing variables to $\overline{r}_c = \frac{1}{2}(\overline{r}' + \overline{r}'')$ and $\overline{r}_d = \overline{r}' - \overline{r}''$ simplifies things considerably
- Also let's modify the limits to reflect a finite area surface (A) rather than infinite area; we need to do this because in the above we would divide by an infinite area to get a cross section per unit area

- Thus making these changes we obtain

$$< |I|^2 > = \int\int_A d\overline{r}_c \int\int_A d\overline{r}_d \; e^{ik_{dx}r_{dx}} e^{ik_{dy}r_{dy}} e^{-(k_{dz}h)^2(1-C(|\overline{r}_{d\perp}|))}$$

where $|\overline{r}_{d\perp}| = \sqrt{r_{dx}^2 + r_{dy}^2}$

- The outer integral just results in A
- The total incoherent power goes as $< |I|^2 > - | < I > |^2$ so we can perform a similar analysis for the coherent power and subtract it from the above to obtain

$$< |I|^2 > - | < I > |^2 = A \int\int_A d\overline{r}_d \; e^{ik_{dx}r_{dx}} e^{ik_{dy}r_{dy}} \left(e^{-(k_{dz}h)^2(1-C(|\overline{r}_{d\perp}|))} - e^{-(k_{dz}h)^2} \right)$$

- Notice since $C(0) = 1$ and $C(\infty) = 0$, the integrand above should become zero for distances bigger than a few correlation lengths. If our surface area is much bigger than a correlation length squared, we can replace the limits with infinity

II. Cross section per unit area

- Now that we've worked out the statistics of I, we can determine the incoherent bistatic cross section per unit area:

$$\sigma_{bh}(\hat{k}_s, \hat{k}_i) = \frac{4\pi r^2 < |\hat{b}_s \cdot \overline{E}_s|^2 >}{A} \qquad (6)$$

where b refers to the scattered polarization $\hat{h}_s$ or $\hat{v}_s$

- Note the factor of A in the denominator, this cancels the A in our expression for the incoherent power in I
- Putting everything together we finally obtain

$$\sigma_{hh}(\overline{k}_s, \overline{k}_i) = \frac{k^2}{\pi} \left[\frac{(1 + \cos\theta_i \cos\theta_s)\cos\phi_s - \sin\theta_i \sin\theta_s}{\cos\theta_i + \cos\theta_s} \right]^2 D_I$$

with a similar expression for σ_{vh} except for a different angular factor in the brackets

- Here D_I results from the incoherent I and is given by

$$D_I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\overline{r}_d\; e^{ik_{dx}r_{dx}} e^{ik_{dy}r_{dy}} \left(e^{-(k_{dz}h)^2(1-C(|\overline{r}_{d\perp}|))} - e^{-(k_{dz}h)^2} \right)$$

- Note our equation for the cross section per unit area is expressed entirely in terms of the surface variance h^2 and the correlation function $C(\rho)$
- However, making progress with the integral for D_I is not possible unless the correlation function is specified. In many cases the integral must be evaluated numerically
- One case for which the integral can be evaluated is the case of a Gaussian correlation function (remember from continuous medium theory) $C(\rho) = e^{-\rho^2/l^2}$ where l is the correlation length of the surface
- In this case, expanding the $e^{-(k_{dz}h)^2(1-C(|\overline{r}_{d\perp}|))}$ in a power series results in a series of terms that can be integrated analytically as in the notes

- The result is then an infinite series instead of an integral; in many cases requires caution in adding up series
- This is nice, but remember this is the answer only for one correlation function; other correlation functions require reconsideration of the integral and probably a numerical integration
- Remember also the Gaussian correlation function is a "single length scale" model as it was in continuous random medium theory
- The integral for D_I can sometimes be a difficult one to evaluate even numerically, because it contains both exponential decay and oscillation; for large heights integrand decays rapidly but not in a simple way
- For these reasons, it is unusual to see true PO results (meaning with D_I) except for Gaussian correlation function surfaces. For other types either small or large height limits of the D_I integral are usually used

III. Small height limit

- If $k_{dz}h << 1$, we can approximate the integral for D_I by expanding $e^{-(k_{dz}h)^2(1-C(|\overline{r}_{d\perp}|))} - e^{-(k_{dz}h)^2} \approx k_{dz}^2 h^2 C(|\overline{r}_{d\perp}|)$
- The integral then becomes

$$D_I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\overline{r}_d\; e^{ik_{dx}r_{dx}} e^{ik_{dy}r_{dy}} k_{dz}^2 h^2 C(|\overline{r}_{d\perp}|)$$

  which is exactly the Fourier transform of the correlation function, proportional to the surface spectral density $W(k_{dx}, k_{dy})$
- Thus PO reproduces Bragg scattering in the small height limit. However, it does not reproduce the same angular functions as the SPM for all polarization quantities
- This led to some controversy but it has been shown that the SPM yields the correct polarization quantities. PO is incorrect due to its failure away from specular direction (really due to an inaccurate boundary condition in the small height limit)

IV. Large height limit

- We can also simplify the integral for D_I in the large height limit $k_{dz}h >> 1$
- Again examining the integrand $e^{-(k_{dz}h)^2(1-C(|\overline{r}_{d\perp}|))} - e^{-(k_{dz}h)^2}$, first neglect the $e^{-k_{dz}^2 h^2}$ second term, and notice the first term falls off extremely rapidly for $|\overline{r}_{d\perp}| > 0$
- Therefore we only need to worry about the behavior of C near zero, approximate with a Taylor series about $|\overline{r}_{d\perp}| = 0$
- It turns out we get $C(\rho) \approx 1 - C''(0)\rho^2/2$ from this because the correlation function at the origin must be symmetric for a stationary surface; can't have a linear term in ρ
- Plugging in this expansion we get

$$D_I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\overline{r}_d\; e^{ik_{dx}r_{dx}} e^{ik_{dy}r_{dy}} e^{-(k_{dz}h)^2 C''(0)|\overline{r}_{d\perp}|^2/2}$$

  which can be integrated analytically if considered in polar coordinates

- The result after the integration is

$$D_I = \frac{2\pi}{k_{dz}^2 h^2 C''(0)} \exp\left(-\frac{k_{d\rho}^2}{2k_{dz}^2 h^2 C''(0)}\right) \quad (7)$$

where $k_{d\rho} = \sqrt{k_{dx}^2 + k_{dy}^2}$

- It turns out that $h^2C''(0)$ is exactly the slope variance of the random process, s^2, i.e. the variance of the surface slope at each point
- Thus we can re-write D_I as

$$\frac{2\pi}{k_{dz}^2 s^2} \exp\left(-\frac{k_{d\rho}^2}{2k_{dz}^2 s^2}\right) \quad (8)$$

- This is the "geometrical optics" limit for rough surface scattering; result can be interpreted as relating the scattering at a particular incidence and scattering angle to the probability of obtaining a surface point tilted so as to produce specular reflection

- This result should apply at very high frequencies where the surface roughness is very large compared to λ
- We still neglect shadowing and multiple scattering effects however
- Very convenient result since no integrals are necessary
- In ocean surface scattering, the "composite" surface model actually combines both GO and tilted SPM results
- GO predictions are used for near specular results. At backscattering this usually means from zero to 20 degrees incidence angle
- Beyond 20 degrees, tilted SPM results are used.
- Slope variances used in GO predictions are those for the "long wave" part of the spectrum