Analysis of Supportable Rates in Symmetric Blocking Wavelength Routers

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***Abstract*— Constructing an $n \times n$ non-blocking wavelength router using $n \times n$ optical cross-connects may be impractical due to certain constraints such as the cost or space limitations. Moreover, in many cases the traffic requirements can be handled without a non-blocking router. In this paper, we study blocking wavelength routers constructed using $x \times x,\ x < n$ optical cross-connects without wavelength conversion. We find the set of rates that can be supported between the input and output fibers of a certain set of symmetric blocking routers. We propose a method to construct blocking routers to achieve any given supportable rate in this region for any given x.**

I. Introduction and Motivation

We study the wavelength division multiplexed optical communication system in Fig. 1. Each input node generates τ wavelengths worth of traffic and each output node listens on τ wavelengths. There are no wavelength converters in the system. The traffic pattern of the system can be described by means of a $n \times n$ matrix R where R_{ij} is the amount of traffic flowing from input i to output j.

Clearly any connection matrix has to have the property that the sum of each row and the sum of each column is less than or equal to τ. We call a non-blocking if it can support any such R. Otherwise we call it blocking.

Figure 2 depicts a nonblocking switch (see [1]) which has a switching core constructed using $\tau\ n \times n$ optical crossconnects (OXCs). An $x \times x$ OXC is a single wavelength device capable of connecting its x input fibers to the x output fibers provided that the connection pattern at any time is a 1-to-1 bipartite matching.

The wavelength router given in Fig. 2 involves τ OXCs each with a size $n \times n$. Next we consider using OXCs of size $x \times x,\ x < n$. An example is illustrated in Fig. 3 for $n = 4$ and $k = 2$. The number, $m = 6$, of OXCs in this scheme is larger than the number, $\tau = 3$, of wavelengths at each fiber. This means, each node has to handle τ wavelengths (e.g., each node has τ transceivers) as is the case for nonblocking routers, whereas there exist $m > \tau$ different wavelengths in the system. On the other hand, the size (2×2) of each OXC is cut down significantly.

One can notice that, in the router given in Fig. 3, each input-output pair is connected through exactly one OXC. Hence, this router requires that $R_{ij} \leq 1$ in addition the row and column sum conditions. Thus, it is blocking.

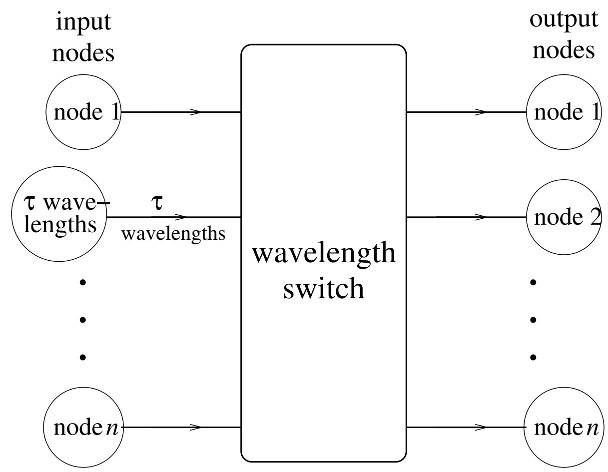

Fig. 1. Each one of the n nodes generate τ wavelength channels worth of traffic, a fraction of which is destined to some other node.

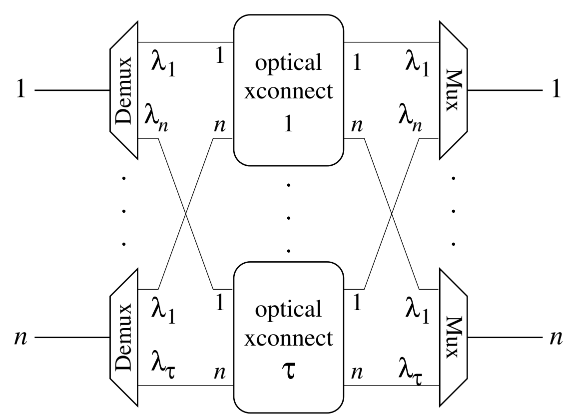

Fig. 2. The non-blocking wavelength router architecture.

In this paper, we study blocking routers, their fundamental limitations and how to build such switches in a systematic way. In particular we focus on the architectures whose switching cores are composed of small OXCs put in parallel. There are a number of motivations for studying such routers.

1) The price of an OXC is proportional to 'crosspoint complexity' (see [2] for a definition). This quantity, on the other hand, turns out to be no less than n^2 for standard architectures.

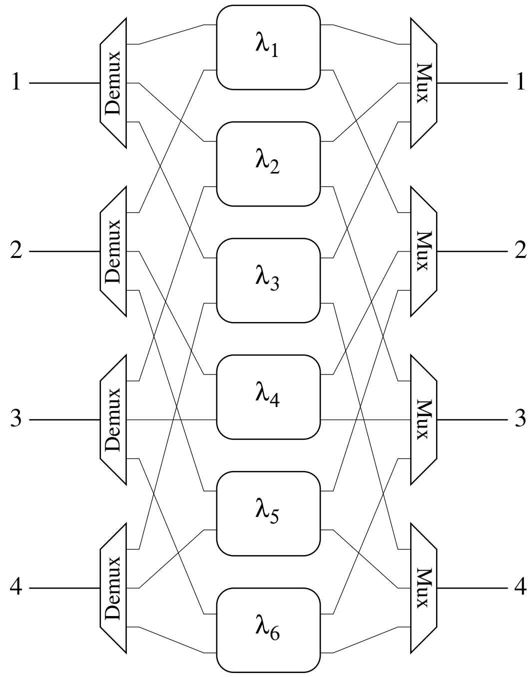

Fig. 3. A blocking wavelength router architecture. The size of the OXCs are smaller compared to the rearrangable version.

2) Even though non-blocking behavior gives tremendous flexibility to a network, it is not always necessary. In many optical networks, the amount of traffic between any given two nodes is no more than a few wavelengths.
3) The $n \times n$ non-blocking wavelength router given in Fig. 2 requires an $n \times n$ OXC per wavelength. The spatial dimensions of an OXC grows almost linearly with its size. It simply may not be feasible to stack an $n \times n$ OXC at a given line card due to space limitations. Under such spatial constraints, smaller OXCs may be preferable.

There are many variations of blocking routers, and in general it is not a trivial task to specify the set of supportable rates for a such routers. We introduce a class of blocking routers and specify the set of rates that are supporable by the router using the theory of majorization ([3]). We illustrate a systematic way to construct blocking routers so that these fundamental limits are approached.

There is a large number of papers on non-blocking wavelength routers and electronic switches (see [4] and [1] for example) and blocking cross-connect architectures (see [2] for a review and [5] for building blocking cross-connects using error control codes). To our knowledge, there is no past work on systematic construction or analysis of blocking wavelength routers.

II. Blocking Wavelength Routers

A typical wavelength router and its parameters are illustrated in Fig. 4 (the internal connections are not shown in this figure.). We denote the number of nodes by n, the number of wavelengths carried at each link by τ, the number of OXCs by m, and the size of each OXC by $x \times x$. It can be seen that

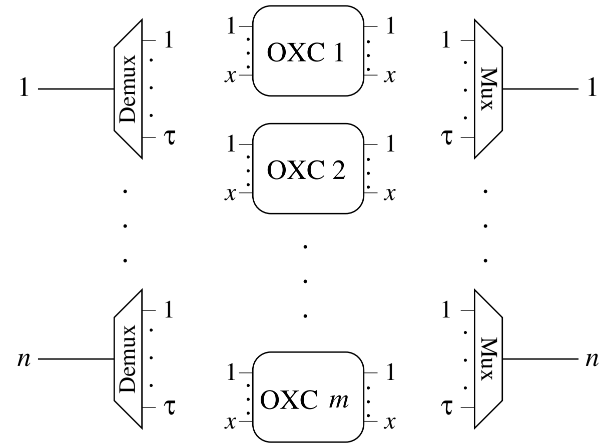

Fig. 4. The blocking router architecture in consideration. Each node is connected to τ of m middle OXCs.

$xm = n\tau$. We assume that the connections between the OXCs and the output nodes are a mirror image of the connections between the OXCs and the input nodes.

In a non-blocking router, there is a separate OXC for each wavelength that a fiber carries and each input (and each output) is connected to every OXC. Hence, $m = \tau$ and $n = x$. In a blocking router, each input and output node is connected to only a subset of the OXCs and vice versa. Thus, $x < \tau < m$.

Compared to a non-blocking architecture the blocking architecture above uses $k = n/x$ times more OXCs, but each OXC has size smaller by a factor $1/k$. Since the crosspoint complexity of a OXC is superlinear in its size, this results in a reduction of crosspoint complexity.

We represent the supportable rates at a node with a vector, $\vec{r}$ of dimension $n-1$, i.e., r_i is the number of existing wavelength connections between an input node and the ith output node different from our node[1]. We call a router *symmetric* if it favors no input node or output node in comparison to others: the set of rates supported by an input (or output) node will be independent of the node identity. Also, for a symmetric router, if a given rate vector, $\vec{r}$, is supportable, then $\vec{r}P$ is also supportable for any permutation matrix P. Moreover, by time sharing, any $\overrightarrow{r'}$ can be supportable so long as it has the form

$$\overrightarrow{r'} = \sum \pi_i \left(\vec{y} P_i \right), \tag{1}$$

where each P_i is a permutation matrix and the coefficients π_i in Eq. 1 constitute a convex combination. Therefore, if some $\vec{r}$ is supportable, any $\overrightarrow{r'}$ in the convex hull of $\vec{r}$ and $(n-1)!-1$ other permutations of $\vec{r}$ is also supportable. Using the theorem by Schur (see [3]), all such $\vec{r'}$ are majorized by $\vec{r}$, i.e., $\vec{r'} \prec \vec{r}$.

[1] Note that we made the input-output node separation for the sake of clarity. In reality, there is a unique node i and two fibers (one input and one output) connect this node to the router. Thus, there is no need for the router to connect input node i to output node i and the available wavelengths are used to communicate with the other $n-1$ nodes. Hence, we define $\vec{r}$ to be $n-1$ dimensional. We assume that the input fiber i and the output fiber i are connected to the same set of OXCs and they carry the same set of wavelengths (even though no OXC connects this pair).

In the next section, we present a theorem on subset partitioning, and based on it we show how to construct a symmetric router for any given size of OXCs, $x \times x$. Next, for symmetric routers, we show that there exists a *maximal vector*, $\vec{w}$ such that if a rate vector, $\vec{r}$ is supportable by a symmetric router, then it satisfies $\vec{r} \prec \vec{w}$. Thus $\vec{w}$ is the boundary for the rates supportable by any symmetric router. We evaluate $\vec{w}$ for symmetric routers as a function of x, m and n. Finally, we prove that a router built using our construction supports $\vec{w}$. Therefore, our construction achieves all the points in this "capacity" region for supportable rates for symmetric routers.

III. Supportable Rates over Blocking Wavelength Routers

In this section, we give the main theorem. Then we illustrate how we choose the triplet, (n, τ, m) and how we arrange the internal connections to construct a symmetric router based on of the theorem.

A. Construction of a Symmetric Router

Theorem 1: Let n, m and x be integers such that $m = \binom{n}{x}$. There exist n subsets of the set $\{1, \ldots, m\}$, such that they all have a cardinality of $\binom{n-1}{x-1}$, and the intersection of any $\chi,\ 2 \leqslant \chi \leqslant x$ of these sets have a cardinality of $\binom{n-\chi}{x-\chi}$.

The proof is constructive and can be found in [6]. Following is an example which gives an intuition on how to construct such n subsets. Suppose $n = 4,\ x = 2$ and $m = \binom{n}{x} = 6$. The following table illustrates the construction of these 4 subsets of $\{1, 2, \ldots, 6\}$. If there is a cross in the (i, j) entry of this table, then the ith subset contains of a j from the set.

	1	2	3	4	5	6
subset 1	×	×	×			
subset 2	×			×	×	
subset 3		×		×		×
subset 4			×		×	×

Each element of $\{1, \ldots, 6\}$ is assigned to every possible 2-combination ($x = 2$) of $n = 4$ sets. Each pair ($k = 2$) of sets has one ($\binom{4}{0}$) element in common. By construction, every pair of these subsets has a distinct common element in $\{1, \ldots, 6\}$. The intersection of every triple is an empty set.

Now, suppose we have a system of $n = 4$ nodes each with $\tau = 3$ transceivers tuned to three of $m = 6$ wavelengths as shown in the above table. Namely, let the numbers $1, \ldots, 6$ (rows) represent the OXCs (distinct wavelengths), the 4 subsets (columns) represent the input (or output) nodes and the crosses represent the internal connections between input (and output) nodes and the OXCs. This blocking router is, in fact, the one illustrated in Fig. 3. Since each pair of nodes share exactly one OXC (i.e., wavelength), each input node can send up to one wavelength of traffic to any given output node. Any point in the region of rates that are supportable by a node satisfies the following:

$$\vec{r} \prec [1\ 1\ 1].$$

Let us go over this construction for routers with OXCs of size $x \times x$ for any given x. First, recall that OXC connections are symmetric for input and output nodes. For instance, if OXC 2 has a connection to the first input node, it is also connected to the first output node. Thus, we shall say, "OXC 2 is connected to the first node," instead of "OXC 2 is connected to both the first input and the first output node." Since x nodes can be connected to each OXC, each column of the table consists of x crosses. In our construction we use $m = \binom{n}{x}$ OXCs. Note that this is the necessary number of OXCs for symmetry. Indeed, if a specific group of x input-output node pairs share a common wavelength, then any group of x input-output node pairs must share a common wavelength. The minimum number of OXCs needed for this is $m = \binom{n}{x}$ (e.g., we need only $m = n$ OXCs if $x = n - 1$). Therefore, our construction is unique in the sense that, there is no other way to construct the router and have symmetry.

Every x combination of the nodes will be matched with exactly one of the OXCs (and a distinct wavelength is assigned to each one of the m OXCs). Thus, in this construction each node carries

$$\tau = \frac{mx}{n} = \binom{n-1}{x-1} \tag{2}$$

wavelengths, which also equals the number of OXCs each node is connected to (e.g., each node carries $\tau = n - 1$ wavelengths if $x = n - 1$). Similarly, to find the number of OXCs (wavelengths) shared by any given pair of nodes we count the number of columns where both of these nodes have a cross. For such columns, all the $x - 2$ combinations of the other $n - 2$ nodes are covered, i.e., any given node pair has $\binom{n-2}{x-2}$ common wavelengths. Also, as a consequence of Theorem 1, any $\chi,\ 2 \leqslant \chi \leqslant x$ nodes have $\binom{n-\chi}{x-\chi}$ common wavelengths.

We can trivially generalize this construction for $m = \kappa\binom{n}{x},\ \kappa \in \mathbb{Z}^+$ such that each one of the n subsets has $\kappa\binom{n}{x}$ elements and the intersection of any given χ of these subsets has a cardinality $\kappa\binom{n-\chi}{x-\chi}$. To achieve this construction, all we have to do is to repeat the same assignment κ times for each group of $\binom{n}{x}$ elements. It can also be seen that the number of distinct wavelength connections also increases by a factor κ with this scaling.

B. Supportable Rates by Symmetric Routers

In this section we find a maximal rate vector $\vec{w}$ which majorizes any rate $\vec{r}$ supportable by the construction. Then, we show that the router we constructed has the property that each input or output node supports any permutation of $\vec{w}$. We start with the symmetric routers with $m = \binom{n}{x}$. The generalization for the case where $\kappa > 1$ will be obvious.

Let us define $\mathcal{M}(\eta; \eta')$ as the set of OXCs that are common to input node η and output node η'. Also let $\eta \neq \eta_j$ for $j,\ 1 \leqslant j \leqslant n - 1$ and define

$$\mathcal{M}(\eta; \eta_1, \eta_2, \ldots, \eta_j) = \bigcup_{l=1}^{j} \mathcal{M}(\eta; \eta_l) \tag{3}$$

Without loss of generality, we consider input node n. What we derive for node n is valid for every node. From basic set theory, for any $\chi, 1 \leqslant \chi \leqslant n-1$,

$$|\mathcal{M}(n;1,2,\ldots,\chi)| = \sum_{j=1}^{\chi} |\mathcal{M}(n;j)| - \sum_{j=1}^{\chi-1}\sum_{k=j+1}^{\chi} |\mathcal{M}(n;j,k)| + \sum_{j=1}^{\chi-2}\sum_{k=j+1}^{\chi-1}\sum_{l=k+1}^{\chi} |\mathcal{M}(n;j,k,l)| - \cdots.$$

For any symmetric router, for $\chi < x-1$,

$$|\mathcal{M}(n;1,\ldots,\chi)| = \binom{\chi}{1}\binom{n-2}{x-2} - \binom{\chi}{2}\binom{n-3}{x-3} + \binom{\chi}{3}\binom{n-4}{x-4} - \cdots \binom{\chi}{\chi}\binom{n-\chi-1}{x-\chi-1}, \quad (4)$$

and for $\chi \geqslant x-1$,

$$|\mathcal{M}(n;1,2,\ldots,\chi)| = \binom{\chi}{1}\binom{n-2}{x-2} - \binom{\chi}{2}\binom{n-3}{x-3} + \cdots \binom{\chi}{x-1}\binom{n-x}{0}. \quad (5)$$

Let $\vec{r}$ be a supportable rate vector and let $\vec{r}_{\downarrow} = \left[r_{[1]}, r_{[2]}, \ldots, r_{[n-1]}\right]$ denote the decreasing rearrangement of the entries of $\vec{r}$. Since $|\mathcal{M}(n;1,\ldots,\chi-1)|$ is the maximum total number of full wavelength connections that could be set up between any given χ input-output node pairs, we can write the following set of inequalities for node n.

$$\begin{aligned} r_{[1]} &\leqslant |\mathcal{M}(n;1)| \\ r_{[1]} + r_{[2]} &\leqslant |\mathcal{M}(n;1,2)| \\ &\vdots \\ \sum_{\chi=1}^{n-1} r_{[\chi]} &\leqslant |\mathcal{M}(n;1,\ldots,n-1)|. \end{aligned} \quad (6)$$

This set of $n-1$ inequalities constitute a set of upper bounds on the number of wavelength connections that could be set up between an input node and the output nodes. Thus, the vector $\vec{w}$, where

$$w_{\chi} = |\mathcal{M}(n;1,\ldots,\chi)| - |\mathcal{M}(n;1,\ldots,\chi-1)| \quad (7)$$

for $\chi \leqslant n-1$ is a maximal vector for the set of supportable rates, i.e., every supportable rate $\vec{r}$ satisfies $\vec{r} \prec \vec{w}$. Next, we show that the router formed by our construction supports $\vec{w}$.

Input node n and output node 1 can be connected through no more than $\binom{n-2}{x-2}$ OXCs $\in \mathcal{M}(n;1)$. Let this pair of nodes use all these OXCs to set up $\binom{n-2}{x-2}$ connections, i.e., $r_1 = |\mathcal{M}(n;1)| = \binom{n-2}{x-2}$ and thus the first relation is satisfied with equality. Next, starting with $\chi = 2$, we assign OXCs

$$\mathcal{M}(n;1,\ldots,\chi) \backslash \mathcal{M}(n;1,\ldots,\chi-1) \quad (8)$$

to connect n and χ for $\chi \leqslant n-1$.

First, (8) constitutes a valid set of assignments since

1) The same OXC is not used to connect an input node to two distict output nodes, i.e., (8) forms disjoint sets for distinct χ values.
2) The number of connections made with this assignment is identical to the total number of wavelengths available at a node, i.e.,

$$\sum_{\chi=1}^{n-1} r_{\chi} = \sum_{\chi=1}^{n-1} |\mathcal{M}(n;1,\ldots,\chi) \backslash \mathcal{M}(n;1,\ldots,\chi-1)| = \binom{n-1}{x-1}. \quad (9)$$

Since $\mathcal{M}(n;1,\ldots,\chi-1) \subset \mathcal{M}(n;1,\ldots,\chi)$,

$$|\mathcal{M}(n;1,\ldots,\chi) \backslash \mathcal{M}(n;1,\ldots,\chi-1)| = |\mathcal{M}(n;1,\ldots,\chi)| - |\mathcal{M}(n;1,\ldots,\chi-1)|.$$

Therefore,

$$w_{\chi} = r_{\chi} \quad (10)$$

for all $\chi \leqslant n-1$, i.e., (6) is satisfied for all $j \leqslant n-1$ with equality. From symmetry, any node can support any permutation of $\vec{w}$.

To summarize, we showed that the vector of number of wavelengths, $\vec{r}$ supported by any given node η satisfies $\vec{r} \prec \vec{w}$. Moreover any $\vec{r}$ that satisfies $\vec{r} \prec \vec{w}$ is supportable by the symmetric router we constructed. Before we finalize the section, let us write $\vec{w}$ explicitly plugging Eq. (4) and (5) into Eq. (7). For any given $x \leqslant n$,

$$\begin{aligned} w_1 &= \binom{n-2}{x-2} \\ w_{\chi} &= \sum_{j=0}^{\min\{\chi-1, x-2\}} (-1)^j \binom{\chi-1}{j}\binom{n-2-j}{x-2-j}. \end{aligned}$$

for all $\chi \leqslant n-1$. For example for $x=2$, $\vec{w} = [1\ 1\ \cdots\ 1]$ and for $x = n-1$, $\vec{w} = [(n-2)\ 1\ 0\ \cdots\ 0]$. Note that for $x = n$, $w_1 = 1$ and $w_{\chi} = 0$ for all $\chi > 1$. This may mislead one to the conclusion that the above formulation fails to calculate the maximal vector for the non-blocking router given in Fig. 2. However, for the non-blocking router $\kappa = n$, hence the maximal vector supportable rates is $\kappa\vec{w} = [n\ 0\ 0\ \cdots\ 0]$.

Another remark we would like to make is that $\vec{w}$ majorizes the vector of number of wavelengths not only from an input node but also to an output node. If we denote the number of wavelength connections between input node i and output node j as the (i,j) entry of a rate matrix R, then each row and each column of R is majorized by $\vec{w}$. Moreover, any rate matrix whose rows and columns are majorized by $\vec{w}$ can be supported by our symmetric router provided that each row and column sum is no more than $\binom{n-1}{x-1}$.

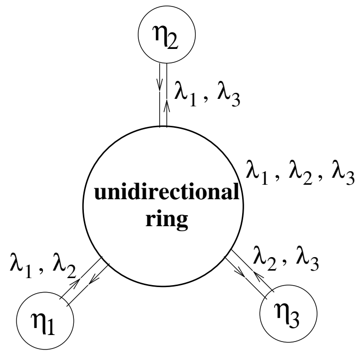

Fig. 5. A three user unilateral optical ring which supports three wavelengths. Every pair of nodes has a distinct common wavelength.

IV. Summary and An Extension

In this paper, we presented a method to construct a symmetric blocking $n \times n$ wavelength router using $x \times x$ OXCs for any given $x < n$. While the crosspoint complexity of this router is $\sim \left(\frac{x}{n}\right)^2$ of its non-blocking couterpart, the maximum number of connections that could be made between any two nodes is reduced by a factor $\frac{x-1}{n-1}$. The ideas we developed this paper can be used for other architectures as well. An example is the ring network, which is one of the most popular architectures for optical networks. Indeed, currently most of the physical layer infrastructure is built around rings.

If a single wavelength is used in a unidirectional ring, only one node can achieve full duplex communication with another node at a time while multiple wavelengths enable many nodes to communicate simultaneously. Using wavelength multiplexers, multiple rings can be supported over the same infrastructure. For any two users to communicate, they must be able to add and drop a common wavelength. Hence, they both need an add-drop multiplexer (ADM) tuned to the same wavelength.

Suppose we have a constraint that each node can have no more than $\tau < n$ ADMs where n is the number of nodes. We believe that an important problem is how to tune these τ ADMs at each node to achieve certain traffic requirements.

If each node uses the same set of τ wavelengths, only a total of τ wavelengths of traffic can be supported. Now, suppose we use a total of $m,\ m > \tau$ wavelengths, and tune each one of τ ADMs of every node similar to that described in Section II for blocking wavelength routers. With this modification, we increased the total amount of traffic (in number of wavelengths) supportable by the ring from τ to m, without changing the cost of the network (no increase in number of ADMs or number of transceivers used at each node). On the other hand, the number of wavelength connections possible between any pair of nodes is reduced to $\tau < n$.

As an example consider the 3-node ring network in Fig. 5. Suppose at most 1 wavelength of traffic is required between each pair of nodes. We can tune the ADMs as illustrated in Fig. 5 so that a total of $m = 3$ wavelength channels of traffic can be supported by the entire network. Each node can add and drop a different pair of wavelengths so that each pair of nodes shares a distinct common wavelength. Thus, at a given time, each node supports $\vec{r} = [1\ 1]$.

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