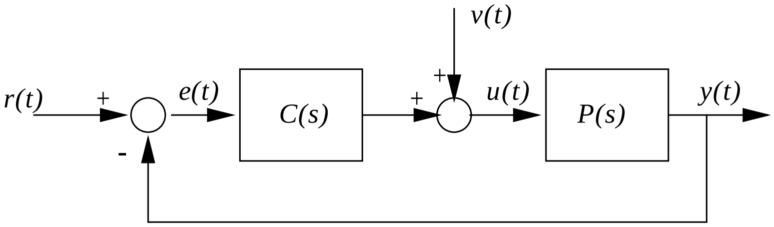
EE 752 Winter 2004 Midterm Exam Solutions

In this examination we deal with the following standard feedback control system, with linear time invariant controller and plant.



Problem 1 (20 points) We know that the main reason to use feedback (as opposed to open loop control) is to reduce the effect of uncertainties. Now suppose that

$$P(s) = \frac{1}{s-3}$$

is known *exactly* (i.e. no uncertainty in the plant), and that the controller parameters can be made precise (i.e. no uncertainty in the controller). Give a reason why we should use feedback control in this case. If there is no reason, explain why.

Answer The most important reason to use feedback in this case is to reduce the effect of $v(t)$, which can be non-zero. In this case, especially since the plant is unstable the output will be unbounded, say for $v(t) =$ unit step, or a sinusoidal signal.

Stabilization is a valid reason here, but if $v(t) = 0$ we can stabilize the system by selecting a controller having a zero at $s = 3$ (no uncertainty in the plant or controller, and $v(t) = 0$ means we can do perfect control).

Problem 2 (30 points) Let the plant transfer function be in the form

$$P(s) = \frac{s+1}{s^3 + a_1 s^2 + a_2 s + a_3}.$$

The parameters are fixed, independent of each other and unknown in the intervals:

$$a_1 \in [0.1\ ,\ 0.5] \qquad a_2 \in [1\ ,\ 2] \qquad a_3 \in [0.2\ ,\ 0.7]\ .$$

Is it possible to find an integral control of the form

$$C(s) = \frac{K}{s}$$

which makes the closed loop system stable for all values of the parameters, a_1, a_2, a_3? If the answer is no, explain why. If the answer is yes, find a value of K which solves this problem.

Answer. Characteristic polynomial is

$$s^4 + a_1 s^3 + a_2 s^2 + (a_3 + K)s + K = 0$$

Routh-Hurwitz test leads to the following table

$$\begin{array}{ccc} 1 & a_2 & K \\ a_1 & a_3 + K & 0 \\ R_3 & K & \\ R_4 & & \\ K & & \end{array}$$

with

$$R_3 = \frac{a_2 a_1 - (a_3 + K)}{a_1}$$

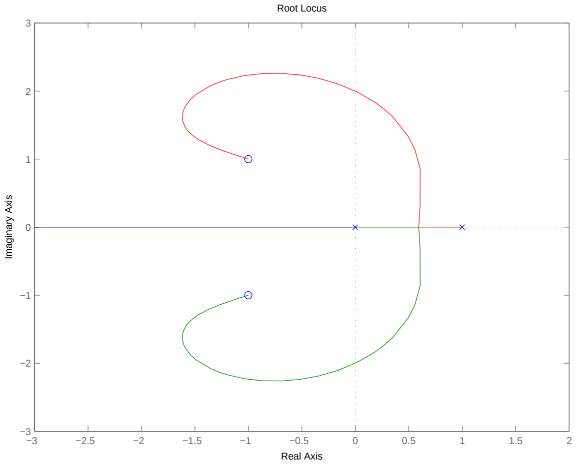
note that we need $K > 0$ and $R_3 > 0$, $R_4 > 0$. Let us examine $R_3 > 0$, condition for the worst case $a_1 = 0.1$, $a_2 = 1$, and $a_3 = 0.7$ note that this leads to $K < -0.6$, which contradicts $K > 0$ condition. So it is impossible to find a K value robustly stabilizing the system.

Problem 3 Given$P(s) = \frac{s^2+2s+2}{s(s-1)}$ and $C(s) = \frac{K}{s}$ with $K > 0$.

(a) (30 points) Sketch the root locus. Note that there is only one break point (you do not need to calculate its exact location). Show other details as much as possible (e.g. Im-axis crossings, and angles of arrivals to complex zeros).

(b) (15 points) Find the values of K for which the real parts of all the closed loop system poles are less than or equal to -1.

(c) (5 points) Find the value of K so that closed loop system poles lie to the left of the line $\text{Re}(s) = -x$, for the largest positive x possible.



Answer. The root locus is shown in the figure

Im-axis crossing can be determined from the R-H test applied to the characteristic equation:

$$s^3 + (K-1)s^2 + 2Ks + 2K = 0$$

this leads $K > 2$ to have a stable feedback system. At $K = 2$ the characteristic polynomial is in the form $(s+\omega^2)(s+r)$ equating this to $s^3 + (2-1)s^2 + 2 \times 2s + 2 \times 2$ we get $\omega = 2$ and $r = 1$. So at $K = 2$ we have roots at $\pm j2$ and -1.

Angle of arrival to complex zero $-1 + j$ is determined from the phase rule

$$2^\circ + (180 - \tan^{-1}\frac{1}{2}) - (\theta + 90^\circ) = 180^\circ$$

this leads to $\theta = 180 - \tan^{-1}\frac{1}{2}$.

To find the K value which leads to two complex poles with real part $-x$ we set the desired characteristic equation to

$$(s^2 + 2xs + \omega^2)(s + r) = s^3 + (2x + r)s^2 + (\omega^2 + 2xr)s + \omega^2 r$$

For part (b) we set $x = 1$ this leads to two solution sets

$$\{\omega^2 = 2, r = \infty, K = \infty\} \quad \text{or} \quad \{\omega^2 = 6, r = 1.5, K = 4.5\}$$

Clearly we take the second set, and in fact for $4.5 < K < \infty$ we have closed loop system poles have real parts less than -1. You could also replace s with $(s-1)$ in the characteristic equation and find the K value this way.

Part (c) requires analytic solution of the above equations in terms of x then we search for the largest possible x that gives a feasible solution. This work is a bit more tedious, and it leads to the following answer:

$$x = \frac{1+\sqrt{5}}{2}, \quad \omega^2 = 3 + \sqrt{5}, \quad r = \omega^2/2, \quad K = r^2$$