ECE600 Midterm Exam #1 Solutions   Autumn 2007

1. *LTI systems and convolution (e.g., homework exercises 1.1, 1.2, 1.3, 1.4, 1.5)*

   (a) System properties

      i. No, not linear. For example, the response to $2x[n]$ is not $2y[n]$, due to the constant offset, +1. (The system is affine.)

      ii. No, not time-invariant. The coefficient $(-0.5)^n$ is time-varying.

      iii. Yes, causal. The system is memory-less.

      iv. No, not BIBO stable. For example, the system grows unbounded for input $\cos(n)$. (The system output does, however, stay bounded for any bounded *and causal* input.)

   (b) For LTI systems in series, the impulse responses convolve. The convolution of $h_s[n] = \{1, 2, 1\}$ and $h_2[n] = \{1, -2, 1\}$ yields

$$h[n] = \begin{cases} 1, & n = 0 \\ 0, & n = 1 \\ -2, & n = 2 \\ 0, & n = 3 \\ 1, & n = 4 \\ 0, & \text{else} \end{cases}$$

   (c) As seen in homework problem 1.4, the step response for causal $h[n]$ is

$$s[n] = \sum_{k=0}^{n} h[k]$$

   Thus we have

$$\begin{aligned} s[0] = h[0] = \delta[0] = 1 \quad &\Rightarrow \quad h[0] = 1 \\ s[1] = h[0] + h[1] = \delta[1] = 0 \quad &\Rightarrow \quad h[1] = 0 - h[0] = -1 \\ s[2] = h[0] + h[1] + h[2] = \delta[2] = 0 \quad &\Rightarrow \quad h[1] = 0 - (h[0] + h[1]) = 0 \\ s[n] = h[0] + h[1] + h[2] + ... + h[n] = \delta[n] = 0 \quad &\Rightarrow \quad h[n] = 0 \text{ for all } n \geq 2 \end{aligned}$$

2. *Difference equations and frequency response (e.g., homework exercises 1.6, 2.2, and 2.3)*

   (a) Recall the general form

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k] \quad \leftrightarrow \quad H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{j\omega k}}$$

   Thus we have

$$H(e^{j\omega}) = \frac{7 + e^{-j3\omega}}{1 - 0.5e^{-j3\omega}}$$

(b) Observe that sinusoidal input input has frequency $\omega = \pi/3$. Note

$$e^{-j3\cdot\pi/3} = e^{-j\pi} = e^{j\pi} = -1$$

Thus we have frequency response

$$H(e^{j\pi/3}) = \frac{7 + (-1)}{1 - 0.5(-1)} = \frac{6}{3/2} = 4\angle 0$$

Use this gain and phase to write the sinusoidal steady-state response

$$y_{ss}[n] = A\,|H(e^{j\pi/3})|\cos\left(\frac{\pi}{3}n + \pi/5 + \angle H(e^{j\pi/3})\right) = 4\cos\left(\frac{\pi}{3}n + \pi/5\right)$$

3. *DTFT (e.g., homework exercises 2.4, 2.5, 2.6, and 2.7)*

(a) The given sequence, $x[n]$ is a pulse of length 7 samples that commmences at time $n = -2$. Recall from exercise 2.6 that the Dirichlet function is the DTFT of the centered pulse; for odd L, we have

$$w[n] = \begin{cases} 1, & -(L-1)/2 \le n \le (L-1)/2 \\ 0, & \text{else} \end{cases} \quad \leftrightarrow \quad \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

Thus, the given $x[n]$ is this Dirichlet function with a delay of one sample. Using the delay property, we have

$$X(e^{j\omega}) = e^{-j\omega}W(e^{j\omega}) = e^{-j\omega}\frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

(b) For time revseral, we have the DTFT relation

$$x[-n] \leftrightarrow X(e^{-j\omega})$$

Specifically, for real-valued time signals, this results in

$$x[-n] \leftrightarrow \text{conj}\left(X(e^{j\omega})\right)$$

Thus, we conjugate the answer from (a) to arrive at our solution:

$$x[-n] \leftrightarrow e^{+j\omega}\frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

4. *Sampling and aliasing (e.g., homework exercises 3.1 – 3.6)*

(a) (iv). We are given at sinusoidal tone at 4 Hz and a sampling rate of 20 Hz. A tone with the same amplitude and a frequency of $4 \pm k20$ Hz for any integer k will yield the same samples. Also, sin is an odd function. Thus, with $T = 1/20$ sec, we have

$$x[n] = 2\sin(2\pi 4nT) = 2\sin(2\pi(4-20)nT) = 2\sin(2\pi(-16)nT) = -2\sin(2\pi 16nT)$$

(b) For bandpass sampling (with a single ADC), we require that the passband of the analog signal fit within a single Nyquist zone. That is,

$$T \le \frac{1}{2}\frac{\lfloor 43/(43-37)\rfloor}{43} = \frac{\lfloor 43/6\rfloor}{86} = \frac{7}{86}\text{ milliseconds}$$

Thus, the minimum required sampling rate is

$$f_s = \frac{1}{T} = \frac{86}{7} = 12\frac{2}{7} \approx 12.28\,\text{kHz}$$